# A First Course in General Relativity - Selected Solutions

Andrew Valentini

September 2023 - Present

## 1 Chapter 1: Special Relativity

## 2 Chapter 2: Vector Analysis in Special Relativity

**Exercise 2.3:** Prove Eq. (2.5)

**Solution:** By convention, Latin indices are not summed over 0 so if we are to interchange them with Greek indices as a dummy index, we must perform the following:

$$\Lambda^{\bar{\alpha}}_{\beta}\Delta x^{\beta} = \Lambda^{\bar{\alpha}}_{0}\Delta x^{0} + \Lambda^{\bar{\alpha}}_{i}\Delta x^{i} \tag{1}$$

since $\Lambda^{\bar{\alpha}}_{\beta}\Delta x^{\beta}$ implies a sum over all of the positive real numbers where $\Lambda^{\bar{\alpha}}_{i}\Delta x^{i}$ is a sum over positive real numbers not including zero.

**Exercise 2.7a:** Prove Eq. (2.10) for all $\alpha, \beta$

**Solution:** To verify that $(\vec{e}_{\alpha})^{\beta} = \delta^{\beta}_{\alpha}$, consider an arbitrary basis vector, $\vec{e}_{\alpha}$, meaning that the elements in its list are all zero except for the single entry at the $\alpha$th component. This can be written as:

$$\vec{e}_{\alpha} = (..., 0, 0, 1, 0, ...) \tag{2}$$

Where the index of each value in the list can be traced with respect to $\alpha$

$$(\alpha - n, ..., \alpha - 2, \alpha - 1, \alpha, \alpha + 1, ..., \alpha + n) \tag{3}$$

Then $(\vec{e}_{\alpha})^{\beta}$ indicates the $\beta$th component of the basis vector $\vec{e}_{\alpha}$. By the definition of a basis vector, we know that all entries in $\vec{e}_{\alpha}$ are zero except the one at the $\alpha$th component. So if we choose $\beta$ to be any non-$\alpha$ index, the result must be 0:

$$(\vec{e}_{\alpha})^{\alpha-1} = 0 \tag{4}$$

It's for this reason that we can define the $\beta$th component of the $\vec{e}_{\alpha}$ basis vector to be equal to the Kronecker delta, meaning that $(\vec{e}_{\alpha})^{\beta} = 1$ only when $\alpha = \beta$.

**Exercise 2.29:** Prove, using component expressions, Eqs. (2.24) and (2.26), that

$$\frac{d}{d\tau}(\vec{U} \cdot \vec{U}) = 2\vec{U} \cdot \frac{d\vec{U}}{d\tau} \tag{5}$$

**Solution:** By (2.26):

$$\begin{aligned} \vec{U} \cdot \vec{U} &= -U^0U^0 + U^1U^1 + U^2U^2 + U^3U^3 \\ &= -(U^0)^2 + (U^1)^2 + (U^2)^2 + (U^3)^2 \end{aligned} \tag{6}$$

and by (2.24):

$$\vec{U} \cdot \vec{U} = \vec{U}^2 \implies \frac{d}{d\tau}(\vec{U} \cdot \vec{U}) = \frac{d}{d\tau}(\vec{U}^2) = 2\vec{U} \cdot \frac{d\vec{U}}{d\tau} \tag{7}$$

# 3 Chapter 3: Tensor Analysis in Special Relativity

**Exercise (3.1a):** Given an arbitrary set of numbers $\{M_{\alpha\beta}; \alpha = 0, ..., 3; \beta = 0, ..., 3\}$ and two arbitrary vector components $\{A^\mu, \mu = 0, ..., 3\}$ and $\{B^\nu, \nu = 0, ..., 3\}$, show that the two expressions

$$M_{\alpha\beta}A^\alpha B^\beta \tag{8}$$

and

$$M_{\alpha\alpha}A^\alpha B^\alpha \tag{9}$$

are not equivalent.

**Solution:**

$$M_{\alpha\alpha}A^\alpha B^\alpha = M_{00}A^0B^0 + M_{11}A^1B^1 + M_1A^1B^1 + M_{11}A^1B^1 \tag{10}$$

where

$$M_{\alpha\beta}A^\alpha B^\beta = B^\beta(M_{0\beta}A^0 + M_{1\beta}A^1 + M_{2\beta}A^2 + M_{3\beta}A^3) \tag{11}$$

So $M_{\alpha\alpha}A^\alpha B^\alpha$ only contains the diagonal terms of $M_{\alpha\beta}A^\alpha B^\beta$.
**Exercise (3.1b):** Show that $A^\alpha B^\beta \eta_{\alpha\beta} = -A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3$

**Solution:**
Because

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{12}$$

Any component of $A^\alpha B^\beta \eta_{\alpha\beta}$ where $\alpha \neq \beta$ means multiplying $A^\alpha B^\beta$ by an off-diagonal component of $\eta_{\alpha\beta}$, which are all 0. Treating $A^\alpha$ and $B^\beta$ as row/column matrices and carrying out their multiplication with $\eta_{\alpha\beta}$ will result in $-A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3$.
**Exercise (3.3a):** Prove, by writing out all of the terms, the validity of the following:

$$\tilde{p}(A^\alpha \vec{e}_\alpha) = A^\alpha \tilde{p}(\vec{e}_\alpha) \tag{13}$$

**Solution:**
Since one-forms act on vector arguments, the scalar values associated with $A^\alpha$ may be pulled out of the expression like so:

$$\begin{aligned} \tilde{p}(A^\alpha \vec{e}_\alpha) = \tilde{p}(A^0\vec{e}^0 + A^1\vec{e}^1 + A^2\vec{e}^2 + A^3\vec{e}^3) = \tilde{p}(A^0\vec{e}_0) + \tilde{p}(A^1\vec{e}_1) + \tilde{p}(A^2\vec{e}_2) + \tilde{p}(A^3\vec{e}_3) \\ = A^0\tilde{p}(\vec{e}_0) + A^1\tilde{p}(\vec{e}_1) + = A^2\tilde{p}(\vec{e}_2) + A^3\tilde{p}(\vec{e}_3) = A^\alpha\tilde{p}(\vec{e}_\alpha) \end{aligned} \tag{14}$$

**Exercise (3.5):** Justify each step leading from Eqs. (3.10a) to (3.10d).

**Solution:** To establish the frame-independence of $A^{\bar{\alpha}}p_{\bar{\alpha}}$:

$$\begin{aligned} A^{\bar{\alpha}}p_{\bar{\alpha}} &= A^{\bar{\alpha}}\tilde{p}(\vec{e}_{\bar{\alpha}}), \\ \vec{e}_{\bar{\alpha}} &= \Lambda^\mu_{\bar{\alpha}}\vec{e}_\mu, \\ A^{\bar{\alpha}} &= \Lambda^{\bar{\alpha}}_\beta A^\beta \\ \implies A^{\bar{\alpha}}\tilde{p}(\vec{e}_{\bar{\alpha}}) = \Lambda^{\bar{\alpha}}_\beta A^\beta \tilde{p}(\Lambda^\mu_{\bar{\alpha}}\vec{e}_\mu) = \Lambda^{\bar{\alpha}}_\beta \Lambda^\mu_{\bar{\alpha}} A^\beta \tilde{p}(\vec{e}_\mu) &= \Lambda^{\bar{\alpha}}_\beta \Lambda^\mu_{\bar{\alpha}} A^\beta p_\mu \end{aligned} \tag{15}$$

and by Eq. (2.18):

$$\Lambda^{\bar{\alpha}}_\beta \Lambda^\mu_{\bar{\alpha}} A^\beta p_\mu = \delta^\mu_\beta A^\beta p_\mu = A^\beta p_\beta \implies A^{\bar{\alpha}}p_{\bar{\alpha}} = A^\beta p_\beta \tag{16}$$

which should not be a surprising result given that the one-form of a vector produces a scalar and scalars are invariant quantities under Lorentz transformations.
**Exercise (3.10a):** Given a frame $\mathcal{O}$ whose coordinates are $\{x^\alpha\}$, show that:

$$\frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha_\beta \tag{17}$$

**Solution:** Taking the partial with respect to $x^\beta$ means holding all terms in $x$ constant except for the $\beta$ index. When

$\alpha \neq \beta$, you are taking a partial of a fixed value (a constant), meaning your derivative will be equal to zero. When you differentiate the $\beta$ index with respect to $\beta$, you will always get 1.

**Exercise (3.10b)**: For any two frames, we have Eq. (3.18):

$$\frac{\partial x^\beta}{\partial x^{\bar{\alpha}}} = \Lambda^\beta_{\bar{\alpha}}. \tag{18}$$

Show that (a) and the chain rule imply

$$\Lambda^\beta_{\bar{\alpha}} \Lambda^{\bar{\alpha}}_\mu = \delta^\beta_\mu \tag{19}$$

**Solution:**

$$\begin{aligned} \Lambda^\beta_{\bar{\alpha}} &= \frac{\partial x^\beta}{\partial x^{\bar{\alpha}}}, \\ \Lambda^{\bar{\alpha}}_\mu &= \frac{\partial x^{\bar{\alpha}}}{\partial x^\mu} \\ \implies \Lambda^\beta_{\bar{\alpha}} \Lambda^{\bar{\alpha}}_\mu &= \frac{\partial x^\beta}{\partial x^{\bar{\alpha}}} \frac{\partial x^{\bar{\alpha}}}{\partial x^\mu} = \frac{\partial x^\beta}{\partial x^\mu} = \delta^\beta_\mu \end{aligned} \tag{20}$$

**Exercise (3.11a):** Use the notation $\partial\phi/\partial x^\alpha = \phi_{,\alpha}$ to rewrite Eqs. (3.14), (3.15), and (3.18).

**Solution:**

Eq. (3.14):

$$\frac{\partial \phi}{\partial t} U^t + \frac{\partial \phi}{\partial t} U^x + \frac{\partial \phi}{\partial t} U^y + \frac{\partial \phi}{\partial t} U^z \implies \phi_{,t} U^t + \phi_{,x} U^x + \phi_{,y} U^y + \phi_{,z} U^z \tag{21}$$

Eq. (3.15):

$$\tilde{d}\phi \underset{\mathcal{O}}{\to} \left( \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \implies \tilde{d}\phi \underset{\mathcal{O}}{\to} (\phi_{,t}, \phi_{,x}, \phi_{,y}, \phi_{,z}) \tag{22}$$

Eq. (3.18):

$$\frac{\partial x^\beta}{\partial x^{\bar{\alpha}}} = \Lambda^\beta_{\bar{\alpha}} \implies x^\beta_{,\bar{\alpha}} = \Lambda^\beta_{\bar{\alpha}} \tag{23}$$

**Exercise (3.13):** Prove, by geometric or algebraic arguments, that $\tilde{d}f$ is normal to surfaces of constant $f$.

**Solution:** If we consider any point $P = (t_0, x_0, y_0, z_0)$ along a parameterized level curve $f(\tau) = \phi(t(\tau), x(\tau), y(\tau), z(\tau) = c$, then we can take the gradient to be as follows:

$$\tilde{d}f = \partial\phi_{,t}\Big|_P \frac{dt}{d\tau}\Big|_{\tau_0} + \partial\phi_{,x}\Big|_P \frac{dx}{d\tau}\Big|_{\tau_0} + \partial\phi_{,y}\Big|_P \frac{dy}{d\tau}\Big|_{\tau_0} + \partial\phi_{,z}\Big|_P \frac{dz}{d\tau}\Big|_{\tau_0} = 0 \tag{24}$$

Since this is also the definition of the dot product between two vectors:

$$\left\langle \partial\phi_{,t}\Big|_P, \partial\phi_{,x}\Big|_P, \partial\phi_{,y}\Big|_P, \partial\phi_{,z}\Big|_P \right\rangle \cdot \left\langle \frac{dt}{d\tau}\Big|_{\tau_0}, \frac{dx}{d\tau}\Big|_{\tau_0}, \frac{dy}{d\tau}\Big|_{\tau_0}, \frac{dz}{d\tau}\Big|_{\tau_0} \right\rangle = 0 \tag{25}$$

whose product is equal to zero, we can say that $\tilde{d}f$ and $f$ are normal to each other at every point along a level curve.

**Exercise (3.14):** Let $\tilde{p} \to_{\mathcal{O}} (1, 1, 0, 0)$ and $\tilde{q} \to_{\mathcal{O}} (-1, 0, 1, 0)$ be two one-forms. Prove, by trying two vectors $\vec{A}$ and $\vec{B}$ as arguments, that $\tilde{p} \otimes q \neq \tilde{\tilde{q}} \otimes \tilde{p}$. Then find the components of $\tilde{p} \otimes \tilde{q}$.

**Solution:** Since a one-form supplied with a vector argument: $\tilde{p}(\vec{A}) = p^\alpha A^\alpha$, we can perform the following operations to show $\tilde{p} \otimes \tilde{q} \neq \tilde{q} \otimes \tilde{p}$:

$$\begin{aligned} \tilde{p} \otimes \tilde{q} &= \tilde{p}(\vec{A})\tilde{q}(\vec{B}) = (A^0 + A^1)(-B^0 + B^2) \\ \tilde{q} \otimes \tilde{p} &= \tilde{q}(\vec{A})\tilde{p}(\vec{B}) = (-A^0 + A^2)(B^0 + B^1) \end{aligned} \tag{26}$$

**Exercise (3.16a):** Prove that $\mathbf{h}_{(s)}$ defined by

$$\mathbf{h}_{(s)}(\vec{A}, \vec{B}) = \frac{1}{2}\mathbf{h}(\vec{A}, \vec{B}) + \frac{1}{2}\mathbf{h}(\vec{B}, \vec{A}) \tag{27}$$

is a symmetric tensor.

**Solution:** From Eq. (3.27), we know a tensor $\mathbf{f}$ is symmetric if:

$$\mathbf{f}(\vec{A},\vec{B}) = \mathbf{f}(\vec{B},\vec{A}) \forall \vec{A},\vec{B} \tag{28}$$

So if $\mathbf{h}$ is to be symmetric, then:

$$\frac{1}{2}\mathbf{h}(\vec{A},\vec{B}) - \frac{1}{2}\mathbf{h}(\vec{B},\vec{A}) = 0 \tag{29}$$

Let $\vec{A} = (A_0, A_1, A_2, A_3)$ and $\vec{B} = (B_0, B_1, B_2, B_3)$ then:

$$\frac{1}{2}\mathbf{h}(\vec{A},\vec{B}) = \frac{1}{2}(A_0B_0 + A_1B_1 + A_2B_2 + A_3B_3) \tag{30}$$

and

$$\frac{1}{2}\mathbf{h}(\vec{B},\vec{A}) = \frac{1}{2}(B_0A_0 + B_1A_0 + B_2A_2 + B_3A_3) \tag{31}$$

and since multiplication is commutative, Eq. 21 must be true, meaning that $\mathbf{h}_{(s)}(\vec{A},\vec{B})$ is a symmetric tensor.

# 4 Chapter 4: Perfect Fluids in Special Relativity

**Exercise (4.7):** Derive Eq. (4.21).

**Solution:** Using the fact that $T^{\alpha\beta} = \rho U^\alpha U^\beta$ derived from Eq. (4.20), we can begin deriving the expressions in Eq. (4.21):

$$\begin{aligned} T^{00} = \rho U^0 U^0 &= \rho \frac{1}{\sqrt{1-v^2}} \frac{1}{\sqrt{1-v^2}} = \frac{\rho}{1-v^2} \\ T^{0i} = \rho U^0 U^i &= \rho \frac{1}{\sqrt{1-v^2}} \frac{v^i}{\sqrt{1-v^2}} = \frac{\rho v^i}{1-v^2} \\ T^{i0} = \rho U^i U^0 &= \frac{\rho v^i}{1-v^2} \\ T^{ij} = \rho U^i U^j &= \frac{\rho v^i v^j}{1-v^2} \end{aligned} \tag{32}$$

**Exercise (4.10):** Take the limit of Eq. (4.35) for $|\vec{V}| << 1$ to get $\partial n/\partial t + \partial(nv^i)/\partial x^i = 0$

**Solution:** Beginning with Eq. (4.35):

$$\frac{\partial}{\partial x^\alpha}\left(nU^\alpha\right) = 0 \tag{33}$$

With the knowledge that $U^\alpha$ contains both spatial components and a temporal component, the previous expression must be separated into two parts:

$$\frac{\partial}{\partial t}\left(nU^t\right) + \frac{\partial}{\partial x^i}\left(nU^i\right) = 0 \tag{34}$$

Using the expressions give on page 93 for $U^t$ and $U^i$, this becomes:

$$\frac{\partial}{\partial t}\left(\frac{n}{\sqrt{1-v^2}}\right) + \frac{\partial}{\partial x^i}\left(\frac{nv^i}{\sqrt{1-v^2}}\right) = 0 \tag{35}$$

Then taking the limit where the speed is much less than 1 makes $1 - v^2 \approx 1$ so this expression becomes:

$$\frac{\partial n}{\partial t} + \frac{\partial(nv^i)}{\partial x^i} = 0 \tag{36}$$

**Exercise (4.12):** Derive Eq. (4.37) from Eq. (4.36)

**Solution:** Eq. (4.36) is given as:

$$T^{\alpha\beta} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{37}$$

So we can prove Eq. (4.37) by checking all of the cases like so:

$$T^{00} = (\rho + p)U^0U^0 + r\eta^{00} \tag{38}$$

In the momentarily comoving reference frame (MCRF), $U^\alpha = (-1, 0, 0, 0)$ so we have:

$$T^{00} = \rho \tag{39}$$

which is what is given in Eq. (3.36). This can be carried out for all cases to find that $T^{0i} = 0, T^{i0} = 0, T^{ii} = p, T^{ij} = 0$ which satisfies Eq. (3.36) so Eq. (4.37) is a valid equation.

**Exercise (4.17):** We have defined $a^\mu = U^\mu_{,\beta} U^\beta$. Go to the relativistic limit (small velocity) and show that $a^i = \dot{v}^i + (\vec{v} \cdot \nabla) v^j = Dv^i/Dt$ where the operator $D/Dt$ is the usual "total" or "advective" time derivative of fluid dynamics.

**Solution:** Writing out an initial expression for $a^i$ while again keeping in mind that $U^i$ contains a temporal component and spatial components:

$$a^i = \frac{\partial U^i}{\partial x^\beta} U^\beta = \frac{\partial U^i}{\partial t} U^t + \frac{\partial U^i}{\partial x^j} U^j = \frac{\partial}{\partial t}\left(\frac{v^i}{\sqrt{1-v^2}}\right)\frac{1}{\sqrt{1-v^2}} + \frac{\partial}{\partial x^j}\left(\frac{v^i}{\sqrt{1-v^2}}\right)\frac{v^j}{\sqrt{1-v^2}} = 0 \tag{40}$$

Taking the non-relativistic limit:

$$a^i = \frac{\partial v^i}{\partial t} + \frac{\partial v^i}{\partial x^j} v^j = 0 \tag{41}$$

And since $\partial v^i / \partial x^j$ is just the dot product between the $i$th velocity component and the spatial derivatives under the Einstein summation convention, this expression becomes:

$$a^i = \dot{v}^i + (\vec{v} \cdot \nabla) v^j \tag{42}$$

Which is an expression defined to be the material or "advective" derivative used in fluid mechanics.

# 5 Chapter 5: Preface to Curvature

**Exercise (5.3a):** Show that the coordinate transformation $(x, y) \to (\xi, \eta)$ with $\xi = x$ and $\eta = 1$ violates Eq. (5.6).

**Solution:** For a transformation to be reasonable, it must assign all coordinates in the source $(x, y)$ to distinct coordinates in the target $(\xi, \eta)$. This property will be satisfied if the Jacobian is non-zero, which is the definition given by Eq. (5.6). So to show this transformation is not reasonable, it must be shown to violate Eq. (5.6):

$$\det\begin{pmatrix} \partial\xi/\partial x & \partial\xi/\partial y \\ \partial\eta/\partial x & \partial\eta/\partial y \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0 \tag{43}$$

So this transformation of coordinates is not reasonable.

**Exercise (5.7):** Calculate all elements of the transformation matrices $\Lambda^{\alpha'}_{\beta}$ and $\Lambda^{\nu'}_{\mu}$ for the transformation from Cartesian $(x, y)$ - the unprimed indices - to polar $(r, \theta)$ - the primed indices.

**Solution:** Since, by Eq. (5.8):

$$\Lambda^{\alpha'}_{\beta} = \begin{pmatrix} \partial\xi/\partial x & \partial\xi/\partial y \\ \partial\eta/\partial x & \partial\eta/\partial y \end{pmatrix} = \begin{pmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial\theta/\partial x & \partial\theta/\partial y \end{pmatrix} \tag{44}$$

We can directly compute the transformation from Cartesian into Polar components by computing the terms of this matrix (knowing that $\xi(x, y) = r = \sqrt{x^2 + y^2}$ and $\eta(x, y) = \theta = \arctan(y/x)$ in polar coordinates). This results in:

$$\Lambda^{\alpha'}_{\beta} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \tag{45}$$

Since $\Lambda^{\nu'}_{\mu}$ is defined as:

$$\Lambda^{\mu}_{\nu'} = \begin{pmatrix} \partial x/\partial\xi & \partial y/\partial\xi \\ \partial x/\partial\eta & \partial y/\partial\eta \end{pmatrix} = \begin{pmatrix} \partial x/\partial r & \partial y/\partial r \\ \partial x/\partial\theta & \partial y/\partial\theta \end{pmatrix} \tag{46}$$

in Eq. (5.13), the matrix is the following:

$$\Lambda^{\mu}_{\nu'} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \tag{47}$$

**Exercise (5.8a):** (Use the result of Exer.7.) Let $f = x^2 + y^2 + 2xy$ and in Cartesian Coordinates $\vec{V} \to (x^2+3y, y^2+3x)$, $\vec{W} \to (1,1)$. Compute $f$ as a function of $r$ and $\theta$, and find the components of $\vec{V}$ and $\vec{W}$ on the polar basis, expressing them as functions of $r$ and $\theta$.

**Solution:** Expressing $f$ as a polar function is as simple as making the substitutions $x = r\cos\theta$ and $y = \sin\theta$, arriving at $f = r^2 + 2r^2\cos\theta\sin\theta$. To express $\vec{V}$ and $\vec{W}$ as polar functions, the same process can be applied. This results in $\vec{V} = (r^2\cos^2\theta + 3r\sin\theta, r^2\sin^2\theta + 3r\cos\theta)$ and $\vec{W} = (1,1)$. To express $\vec{V}$ and $\vec{W}$ in a polar *basis*, though, you must use the transformations found in the previous problem:

$$V^{\alpha'} = \Lambda^{\alpha'}_{\beta}V^{\beta} \implies \vec{V} = \begin{pmatrix} \cos\theta & \sin\theta \\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} r^2\cos^2\theta + 3r\sin\theta \\ r^2\sin^2\theta + 3r\cos\theta \end{pmatrix} = \begin{pmatrix} r^2(\cos^3\theta + \sin^3\theta) + 6r\sin\theta\cos\theta \\ r(\cos\theta\sin^2\theta - \cos^2\theta\sin\theta) + 3(\cos^2\theta - \sin^2\theta) \end{pmatrix} \tag{48}$$

$$\vec{W} = \begin{pmatrix} \cos\theta & \sin\theta \\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta + \sin\theta \\ (\cos\theta - \sin\theta)/r \end{pmatrix} \tag{49}$$

**Exercise (5.8b):** Find the components of $\tilde{d}f$ in Cartesian Coordinates and obtain them in polars (i) by direct calculation in polars, and (ii) by transforming components from Cartesian.

**Solution:** (i) To compute by direct calculation in polar: $\tilde{d}f = (\partial f/\partial r, \partial f/\partial\theta)$ we can use the definition of $f$ in polar that was derived in part (a):

$$\frac{\partial f}{\partial r} = \frac{\partial}{\partial r}\left(r^2 + 2r^2\cos\theta\sin\theta\right) = 2r + 4r\cos\theta\sin\theta \tag{50}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial}{\partial \theta}\left(r^2 + 2r^2\cos\theta\sin\theta\right) = 2r^2\cos(2\theta) \tag{51}$$

(ii) To compute $\tilde{d}f$ by transforming components from Cartesian,

$$\frac{\partial\phi}{\partial\xi} = \frac{\partial x}{\partial\xi}\frac{\partial\phi}{\partial x} + \frac{\partial y}{\partial\xi}\frac{\partial\phi}{\partial y} \implies \frac{\partial f}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial f}{\partial y} \tag{52}$$

$$\frac{\partial f}{\partial r} = \cos\theta(2x+2y) + \sin\theta(2x+2y) = (\cos\theta + \sin\theta)(2r\cos\theta + 2r\sin\theta) = 2r + 4r\sin\theta\cos\theta \tag{53}$$

Similarly:

$$\frac{\partial f}{\partial \theta} = \frac{\partial x}{\partial\theta}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial f}{\partial y} \tag{54}$$

$$\frac{\partial f}{\partial \theta} = (-r\sin\theta)(2x+2y) + (r\cos\theta)(2x+2y) = 2r^2\cos\theta \tag{55}$$

It should be noted that the expressions from (ii) match those derived from (i).

**Exercise (5.8c):** (i) Use the metric tensor in polar coordinates to find the polar components of the one-forms $\tilde{V}$ and $\tilde{W}$ associated with $\vec{V}$ and $\vec{W}$. (ii) Obtain the polar components of $\tilde{V}$ and $\tilde{W}$ by transformation of their Cartesian components.

**Solution:** (i) By Eq. (5.31), the metric tensor in polar coordinates is:

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{56}$$

The metric in polar coordinates can be used to find the polar components of the one-forms by:

$$\tilde{W}_\alpha = g_{\alpha\beta}W^\beta = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \begin{pmatrix} \cos\theta + \sin\theta \\ (\cos\theta - \sin\theta)/r \end{pmatrix} = \begin{pmatrix} \cos\theta + \sin\theta \\ r(\cos\theta - \sin\theta) \end{pmatrix} \tag{57}$$

The same can be done for $\vec{V}$ as computed in polar form from part (a) of this problem.

(ii) Using the transformation matrix $\Lambda^{\alpha}_{\beta'}$ to obtain $\tilde{V}$ and $\tilde{W}$:

$$\Lambda^{\alpha}_{\beta'}\tilde{W}^\alpha = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta + \sin\theta \\ r(\cos\theta - \sin\theta) \end{pmatrix} \tag{58}$$

And the same process can be employed to solve for $\tilde{V}$. Note that $\tilde{V}$ and $\tilde{W}$ is just $\vec{V}$ and $\vec{W}$ in Cartesian coordinates since the metric tensor in Cartesian coordinates is the identity matrix.

**Exercise (5.11a):** For the vector field $\vec{V}$ whose Cartesian components are $(x^2+3y, y^2+3x)$, compute $V^\alpha_{,\beta}$ in Cartesian.

**Solution:** Since $V^\alpha_{,\beta} \equiv \partial V^\alpha/\partial x^\beta$:

$$V^\alpha_{,\beta} = \begin{pmatrix} \partial V^1/\partial x & \partial V^1/\partial y \\ \partial V^2/\partial x & \partial V^2/\partial y \end{pmatrix} = \begin{pmatrix} 2x & 3 \\ 3 & 2y \end{pmatrix} \tag{59}$$

**Exercise (5.11b):** Compute the transformation $\Lambda^{\mu'}_{\alpha}\Lambda^{\beta}_{\nu'}V^\alpha_{,\beta}$ to polars.

This computation is a straightforward usage of the transformation matrices $\Lambda^{\mu'}_{\alpha}$ and $\Lambda^{\beta}_{\nu'}$ from Cartesian to polar coordinates derived in Exercise 5.7 and the polar form of $V^\alpha_{,\beta}$ found in the previous part to this problem. The order of multiplication for these matrices should be noted, however, since computing $\Lambda^{\mu'}_{\alpha}\Lambda^{\beta}_{\nu'}V^\alpha_{,\beta}$ would leave $V^\alpha_{,\beta}$ unchanged. Computing $\Lambda^{\mu'}_{\alpha}V^\alpha_{,\beta}\Lambda^{\beta}_{\nu'}$ results in:

$$\begin{aligned}\Lambda^{\mu'}_{\alpha}V^\alpha_{,\beta}\Lambda^{\beta}_{\nu'} &= \begin{pmatrix} \cos\theta & \sin\theta \\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix}\begin{pmatrix} 2r\cos\theta & 3 \\ 3 & 2r\sin\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \\ &= \begin{pmatrix} 2(r\cos^3\theta + 3\cos\theta\sin\theta + r\sin^3\theta) - r(\cos\theta-\sin\theta) & (-3\sin\theta+\cos\theta)(-3+2r\sin\theta) \\ \frac{(\cos\theta-\sin\theta)(3\cos\theta+3\sin\theta-r\sin(2\theta))}{r} & (-3+r\cos\theta+r\sin\theta)\sin(2\theta) \end{pmatrix}\end{aligned} \tag{60}$$

**Exercise (5.11c):** Compute the components $V^{\mu'}_{;\nu'}$ directly in polars using the Christoffel symbols.

**Solution:** Since $\alpha, \beta \in \{x, y\}$ in Cartesian coordinates, there will be four components to compute: $V^r_{;r}, V^\theta_{;r}, V^r_{;\theta}, V^\theta_{;\theta}$. Beginning with $V^r_{;r}$:

$$\begin{aligned} V^r_{;r} = \frac{\partial V^r}{\partial r} + V^\mu\Gamma^r_{\mu r} \;\&\; \Gamma^\mu_{rr} = \forall\mu &\implies V^r_{;r} = \frac{\partial V^r}{\partial r} + V^\theta\Gamma^r_{\theta r} \\ \Gamma^r_{\theta r} : \frac{\partial\vec{e_\theta}}{\partial r} = -\sin\theta\vec{e_x} + \cos\theta\vec{e_y} = \frac{1}{r}\vec{e_\theta} &\implies \Gamma^r_{\theta r} = 0 \\ \implies V^r_{;r} = \frac{\partial V^r}{\partial r} = \frac{\partial}{\partial r}\left(r^2(\cos^3\theta+\sin^3\theta)+6r\sin\theta\cos\theta\right) &= 2r(\cos^3\theta+\sin^3\theta)+6\sin\theta\cos\theta \end{aligned} \tag{61}$$

$$\begin{aligned} V^\theta_{;r} = \frac{\partial V^\theta}{\partial r} + V^\mu\Gamma^\theta_{\mu r} = \frac{\partial V^\theta}{\partial r} + \frac{1}{r}V^\theta = \\ \frac{\partial}{\partial r}\left(r(\cos\theta\sin^2\theta - \cos^2\theta\sin\theta) + 3(\cos^2\theta-\sin^2\theta)\right) + \frac{1}{r}\left(r(\cos\theta\sin^2\theta-\cos^2\theta\sin\theta)+3(\cos^2\theta-\sin^2\theta)\right) \\ \implies V^\theta_{;r} = \frac{(\cos\theta-\sin\theta)(3\cos\theta+3\sin\theta)-r\sin(2\theta)}{r} \end{aligned} \tag{62}$$

$$\begin{aligned} V^r_{;\theta} = \frac{\partial V^r}{\partial\theta} + V^\mu\Gamma^r_{\mu\theta} = \frac{\partial V^r}{\partial\theta} - rV^\theta \\ \implies V^r_{;\theta} = -r(\cos\theta-\sin\theta)(-3\cos\theta+3\sin\theta)+r\sin(2\theta) \end{aligned} \tag{63}$$

$$V^\theta_{;\theta} = \frac{\partial V^\theta}{\partial\theta} + \frac{1}{r}V^r = \sin(2\theta)(-3+r\cos\theta+r\sin\theta)) \tag{64}$$

**Exercise (5.11d):** Compute the divergence $V^\alpha_{,\alpha}$ using results from part (a).

**Solution:**

$$V^\alpha_{,\alpha} = \frac{\partial V^\alpha}{\partial x^\alpha} = \frac{\partial V^x}{\partial x} + \frac{\partial V^y}{\partial y} = 2(x+y) = 2r(\cos\theta+\sin\theta) \tag{65}$$

**Exercise (5.11e):** Compute the divergence $V^{\mu'}_{;\mu'}$ using results from either part (b) or (c).

**Solution:**

$$\begin{aligned} V^{\mu'}_{;\mu'} = V^r_{;r} + V^\theta_{;\theta} &= \frac{\partial V^r}{\partial r} + \Gamma^r_{rr}V^r + \Gamma^r_{\theta r}V^\theta + \frac{\partial V^r}{\partial r} + \Gamma^\theta_{\theta\theta}V^\theta + \Gamma^\theta_{r\theta}V^r \\ &= \frac{\partial V^r}{\partial r} + \frac{\partial V^\theta}{\partial\theta} + \frac{1}{r}V^r = 2r(\cos\theta+\sin\theta) \end{aligned} \tag{66}$$

**Exercise (5.11f):** Compute the divergence $V^{\mu'}_{;\mu'}$ using Eq. (5.55) directly.

$$V^{\mu'}_{;\mu'} = \frac{1}{r}\frac{\partial}{\partial r}(rV^r) + \frac{\partial}{\partial\theta}V^\theta = 2r(\cos\theta+\sin\theta) \tag{67}$$

(This and the majority of the results given for Exercise 5.11 were computed in Mathematica)

**Exercise (5.12a):** For the one-form field $\tilde{p}$ whose Cartesian coordinates are $(x^2+3y, y^2+3x)$, compute $p_{\alpha,\beta}$ in Cartesian.

**Solution:**

$$p_{\alpha,\beta} = \begin{pmatrix} p_{rr} & p_{r\theta} \\ p_{\theta r} & p_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 2x & 3 \\ 3 & 2y \end{pmatrix} = \begin{pmatrix} 2r\cos\theta & 3 \\ 3 & 2r\sin\theta \end{pmatrix} \tag{68}$$

**Exercise (5.12b):** Compute the transformation $\Lambda^{\alpha}_{\mu'}\Lambda^{\beta}_{\nu'}p_{\alpha,\beta}$ to polars.

**Solution:**

$$\Lambda^{\alpha}_{\mu'}\Lambda^{\beta}_{\nu'}p_{\alpha,\beta} = (\Lambda^{\alpha}_{\mu'})^T p_{\alpha,\beta}\Lambda^{\beta}_{\nu'} = \begin{pmatrix} 2r(\cos^3\theta - 3\cos\theta\sin\theta + r^2\sin^3\theta) & \end{pmatrix} \tag{69}$$

**Exercise (5.12c):** Compute the components $p_{\mu';\nu'}$ directly in polars using the Christoffel symbols, Eq. (4.44), in Eq. (5.62).

**Solution:**

$$p_{r;r} = p_{r,r} - p_{\mu}\Gamma^{\mu}_{\alpha\beta} = \frac{\partial p_r}{\partial r} - p_r\Gamma^{r}_{rr} - p_\theta\Gamma^{\theta}_{rr} \implies p_{r;r} = \frac{\partial p_r}{\partial r} \tag{70}$$

Where $p_r$ is the $r$-component of the one-form in a polar basis.

$$p_r = r^2(\cos^3\theta + \sin^3\theta) + 6r\sin\theta\cos\theta \implies p_{r;r} = \frac{\partial p_r}{\partial r} = 2r(\cos^3\theta + \sin^3\theta) + 6\sin\theta\cos\theta \tag{71}$$

$$p_{r;\theta} = \frac{\partial p_r}{\partial\theta} - p_r\Gamma^{r}_{r\theta} - p_\theta\Gamma^{\theta}_{r\theta} = \frac{\partial p_r}{\partial\theta} - \frac{1}{r}p_\theta \tag{72}$$

**Exercise (5.14):** For the tensor whose polar coordinates are $(A^{rr} = r^2, A^{r\theta} = r\sin\theta, A^{\theta r} = r\cos\theta, A^{\theta\theta} = \tan\theta)$, compute in Eq. (5.65) in polars for all possible indices:

$$\begin{aligned} \nabla_r A^{rr} = \frac{\partial A^{rr}}{\partial r} + A^{\alpha r}\Lambda^{r}_{\alpha r} + A^{r\alpha}\Lambda^{r}_{\alpha r} &= \frac{\partial A^{rr}}{\partial r} + A^{rr}\Lambda^{r}_{rr} + A^{\theta r}\Lambda^{r}_{\theta r} + A^{rr}\Lambda^{r}_{rr} + A^{r\theta}\Lambda^{r}_{\theta r} \\ &\implies \nabla_r A^{rr} = \frac{\partial A^{rr}}{\partial r} = \frac{\partial}{\partial r}\left(r^2\right) = 2r \end{aligned} \tag{73}$$

$$\begin{aligned} \nabla_\theta A^{rr} &= \frac{\partial A^{rr}}{\partial\theta} + A^{rr}\Lambda^{r}_{r\theta} + A^{\theta r}\Lambda^{r}_{\theta\theta} + A^{rr}\Lambda^{r}_{r\theta} + A^{r\theta}\Lambda^{r}_{\theta\theta} \\ \implies \nabla_\theta A^{rr} = -r\left(A^{\theta r} + A^{r\theta}\right) + \frac{\partial A^{rr}}{\partial\theta} &= -r(r\cos\theta + r\sin\theta) + \frac{\partial}{\partial\theta}\left(r^2\right) = -r^2(\cos\theta + \sin\theta) \end{aligned} \tag{74}$$

$$\begin{aligned} \nabla_\theta A^{r\theta} = \frac{\partial A^{r\theta}}{\partial\theta} + A^{r\theta}\Gamma^{r}_{r\theta} + A^{\theta\theta}\Gamma^{r}_{\theta\theta} + A^{rr}\Gamma^{\theta}_{r\theta} + A^{r\theta}\Gamma^{\theta}_{\theta\theta} &= \frac{\partial A^{r\theta}}{\partial\theta} - r(A^{\theta\theta}) + \frac{1}{r}(A^{rr}) \\ &\implies \nabla_\theta A^{r\theta} = r(\cos\theta - \tan\theta - 1) \end{aligned} \tag{75}$$

And the five remaining computations for all possible indices $(\nabla_r A^{r\theta}, \nabla_r A^{\theta r}, \nabla_\theta A^{\theta r}, \nabla_r A^{\theta\theta}, \nabla_\theta A^{\theta\theta})$ can be computed in exactly the same manner.

**Exercise (5.16):** Fill in all the missing steps leading from Eq. (5.74) to Eq. (5.75).

**Solution:** Starting with Eq. (5.72):

$$g_{\alpha\beta;\mu} = g_{\alpha\beta,\mu} - \Gamma^{\nu}_{\alpha\mu}g_{\nu\beta} - \Gamma^{\nu}_{\beta\mu}g_{\alpha\nu} \tag{76}$$

And using the fact that $g_{\alpha'\mu';\beta'} = 0$:

$$g_{\alpha'\beta',\mu'} = \Gamma^{\nu'}_{\alpha'\mu'}g_{\nu'\beta'} + \Gamma^{\nu'}_{\beta'\mu'}g_{\alpha'\nu'} \implies g_{\alpha\beta,\mu} = \Gamma^{\nu}_{\alpha\mu}g_{\nu\beta} + \Gamma^{\nu}_{\beta\mu}g_{\alpha\nu} \tag{77}$$

And since $\alpha, \beta, \mu$ are dummy indices whose order can be rearranged in the previous expression, the following form can be arrived at by switching the $\beta$ and $\mu$ indices:

$$g_{\alpha\mu,\beta} = \Gamma^{\nu}_{\alpha\beta}g_{\nu\mu} + \Gamma^{\nu}_{\mu\beta}g_{\alpha\nu} \tag{78}$$

And the following expression can be arrived at by switching $\alpha$ with $\beta$ in Eq. (78) and multiplying the whole expression by a negative sign:

$$g_{\beta\mu,\alpha} = \Gamma^{\nu}_{\beta\alpha}g_{\nu\mu} + \Gamma^{\nu}_{\mu\alpha}g_{\beta\nu} \implies -g_{\beta\mu,\alpha} = -\Gamma^{\nu}_{\beta\alpha}g_{\nu\mu} - \Gamma^{\nu}_{\mu\alpha}g_{\beta\nu} \tag{79}$$

We can now consider the addition of the three terms, $g_{\alpha\beta,\mu}, g_{\alpha\mu,\beta}, -g_{\beta\mu,\alpha}$:

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = \Gamma^{\nu}_{\alpha\mu} g_{\nu\beta} + \Gamma^{\nu}_{\beta\mu} g_{\alpha\nu} + \Gamma^{\nu}_{\alpha\beta} g_{\nu\mu} + \Gamma^{\nu}_{\mu\beta} g_{\alpha\nu} - \Gamma^{\nu}_{\beta\alpha} g_{\nu\mu} - \Gamma^{\nu}_{\mu\alpha} g_{\beta\nu} \tag{80}$$

And, using the fact that the indices of the metric can be interchanged ($g_{\beta\nu} = g_{\nu\beta}$), we arrive at:

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = \left(\Gamma^{\nu}_{\alpha\mu} - \Gamma^{\nu}_{\mu\alpha}\right) g_{\nu\beta} + \left(\Gamma^{\nu}_{\alpha\beta} - \Gamma^{\nu}_{\beta\alpha}\right) g_{\nu\mu} + \left(\Gamma^{\nu}_{\beta\mu} + \Gamma^{\nu}_{\mu\beta}\right) g_{\alpha\nu} \tag{81}$$

Since the lower indices of the Christoffel symbols may be interchanged, this leaves us with:

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = 2\Gamma^{\nu}_{\beta\mu} g_{\alpha\nu} \tag{82}$$

Using the fact that inverting the metric just turns its covariant indices into contravariant indices ($1/g_{\alpha\beta} = g^{\alpha\beta}$):

$$\Gamma^{\nu}_{\beta\mu} = \frac{1}{2} g^{\alpha\nu} \left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}\right) \tag{83}$$

It's important to remind the reader of the notation being used here to understand the meaning of this result. Recall that $\phi_{,\alpha} \equiv \frac{\partial \phi}{\partial x^{\alpha}}$ so the previous expression becomes:

$$\Gamma^{\nu}_{\beta\mu} = \frac{1}{2} g^{\alpha\nu} \left(\frac{\partial g_{\alpha\beta}}{\partial x^{\mu}} + \frac{\partial g_{\alpha\mu}}{\partial x^{\beta}} - \frac{\partial g_{\beta\mu}}{\partial x^{\alpha}}\right) \tag{84}$$

Meaning the Christoffel symbols can be written in terms of derivatives of the metric.

**Exercise (5.18):** Verify Eq. (5.78).

**Solution:** Since we are working in polar coordinates, $\vec{e}_{\hat{\alpha}} \cdot \vec{e}_{\hat{\beta}}$ can only take on the forms $\vec{e}_r \cdot \frac{1}{r}\vec{e}_\theta$, $\vec{e}_r \cdot \vec{e}_r$, $\frac{1}{r}\vec{e}_\theta \cdot \frac{1}{r}\vec{e}_\theta$, or $\frac{1}{r}\vec{e}_\theta \cdot \vec{e}_r$. Since $\vec{e}_\theta$ will always be orthogonal to $\vec{e}_r$, meaning that $\vec{e}_r \cdot \vec{e}_\theta = \vec{e}_\theta \cdot \vec{e}_r = 0$. This also tells us that $\vec{e}_\theta \cdot \vec{e}_\theta = \vec{e}_r \cdot \vec{e}_r = 1$, satisfying the first part of Eq. (5.78):

$$\vec{e}_{\hat{\alpha}} \cdot \vec{e}_{\hat{\beta}} \equiv g_{\hat{\alpha}\hat{\beta}} = \delta_{\hat{\alpha}\hat{\beta}} \tag{85}$$

A similar argument can be made to prove the second half of Eq. (5.78) since the basis differentials $\tilde{d}r$ and $\tilde{d}\theta$ are orthogonal to each other, resulting in

$$\tilde{\omega}^{\hat{\alpha}} \cdot \tilde{\omega}^{\hat{\beta}} \equiv g^{\hat{\alpha}\hat{\beta}} = \delta^{\hat{\alpha}\hat{\beta}} \tag{86}$$

being a verified statement.

**Exercise (5.22):** Show that if $U^{\alpha}\nabla_{\alpha}V^{\beta} = W^{\beta}$, then $U^{\alpha}\nabla_{\alpha}V_{\beta} = W_{\beta}$

**Solution:** Recall the notation that $\nabla_{\alpha}V^{\beta} \equiv V^{\beta}_{;\alpha}$ from Eq. (5.51). This turns the expression into:

$$U^{\alpha} V^{\beta}_{;\alpha} = W^{\beta} \tag{87}$$

We can then multiply both sides of the expression by the metric $g_{\mu\beta}$:

$$U^{\alpha} g_{\mu\beta} V^{\beta}_{;\alpha} = g_{\mu\beta} W^{\beta} \tag{88}$$

From Eq. (5.68), $V_{\alpha;\beta} = g_{\alpha\mu} V^{\alpha}_{;\beta}$ so we can transform the left hand side of this expression to be:

$$U^{\alpha} V_{\mu;\alpha} = g_{\mu\beta} W^{\beta} \tag{89}$$

And we can finally use $V_{\alpha} = g_{\alpha\mu} V^{\mu}$ from Eq. (5.69) to simply the right side of the expression into:

$$U^{\alpha} V_{\mu;\alpha} = W_{\mu} \tag{90}$$

And since $\mu$ is just a dummy index, it can be changed for $\beta$, resulted in the desired expression:

$$U^{\alpha} \nabla_{\alpha} V_{\beta} = W_{\beta} \tag{91}$$

# 6 Chapter 6: Curved Manifolds

**Exercise (6.6):** Prove that the first term in Eq. (6.37) vanishes.

**Solution:** Starting with Eq. (6.37):

$$\Gamma^{\alpha}_{\mu\alpha} = \frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\alpha} - g_{\mu\alpha,\beta}) + \frac{1}{2}g^{\alpha\beta}g_{\alpha\beta,\mu} \tag{92}$$

We want to show that $g^{\alpha\beta}g_{\beta\mu,\alpha} = g^{\alpha\beta}g_{\mu\alpha,\beta}$ to show that the first term vanishes. To do this, first use the fact that the metric is symmetric so the its indices can be interchanged as in following step:

$$g^{\alpha\beta}g_{\beta\mu,\alpha} \implies g^{\beta\alpha}g_{\beta\mu,\alpha} \tag{93}$$

Then use the fact that the indices considered are dummy indices so the swap $\alpha \to \beta$ & $\beta \to \alpha$ can be made:

$$g^{\beta\alpha}g_{\beta\mu,\alpha} \implies g^{\alpha\beta}g_{\alpha\mu,\beta} \tag{94}$$

And again use the fact that the metric is symmetric so the indices $\alpha$ and $\mu$ can be interchanged:

$$g^{\alpha\beta}g_{\alpha\mu,\beta} \implies g^{\alpha\beta}g_{\mu\alpha,\beta} \tag{95}$$

Reaching the desired result.

**Exercise (6.8):** Fill in the missing algebra leading to Eqs. (6.40) and (6.42).

**Solution:** Starting from Eq. (6.38):

$$\Gamma^{\alpha}_{\mu\alpha} = \frac{1}{2}g^{\alpha\beta}g_{\alpha\beta,\mu} \tag{96}$$

We can swap the $\alpha$ and $\beta$ indices on $g_{\alpha\beta,\mu}$ so that we can use Eq. (6.39):

$$g_{,\mu} = gg^{\alpha\beta}g_{\beta\alpha,\mu} \implies g_{,\mu}\frac{1}{g} = g^{\alpha\beta}g_{\beta\alpha,\mu} \tag{97}$$

as a substitution:

$$\Gamma^{\alpha}_{\mu\alpha} = \frac{1}{2}\frac{1}{g}g_{,\mu} \tag{98}$$

And in a step that I really don't understand, this becomes:

$$\Gamma^{\alpha}_{\mu\alpha} = \frac{1}{\sqrt{-g}}(\sqrt{-g})_{\mu} \tag{99}$$

resulting in Eq. (6.40).

With this, Eq. (5.49) can be used where $\beta \to \alpha$ since this is just a dummy index and using the new definition of $\Gamma^{\alpha}_{\mu\alpha}$ in Eq. (6.40):

$$V^{\alpha}_{;\alpha} = V^{\alpha}_{,\alpha} + V^{\mu}\Gamma^{\alpha}_{\mu\alpha} = V^{\alpha}_{,\alpha} + V^{\mu}\left(\frac{1}{\sqrt{-g}}(\sqrt{-g})_{\mu}\right) \tag{100}$$

$V^{\alpha}_{,\alpha}$ can then be multiplied by $\frac{\sqrt{-g}}{\sqrt{-g}}$ so that it can be combined with the other term in the expression:

$$V^{\alpha}_{;\alpha} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}V^{\alpha}_{,\alpha} + V^{\alpha}\sqrt{-g}_{\mu}\right) \tag{101}$$

And since the term in the parentheses is just the definition of the chain rule:

$$V^{\alpha}_{;\alpha} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}V^{\alpha}\right)_{\alpha} \tag{102}$$

Which results in Eq. (6.42).

**Exercise (6.13a):** Show that if $\vec{A}$ and $\vec{B}$ are parallel-transported along a curve, then $\boldsymbol{g}(\vec{A},\vec{B}) = \vec{A}\cdot\vec{B}$ is constant on the curve.

**Solution:** If we parallel transport $\vec{A}$ and $\vec{B}$ along a curve, $\vec{U}$, then the following condition is satisfied:

$$\frac{d\vec{V}}{d\lambda} = \frac{\partial V^{\alpha}}{\partial x^{\beta}}\frac{dx^{\beta}}{d\lambda} = 0 \tag{103}$$

for both $\vec{A}$ and $\vec{B}$ where $\frac{dx^\beta}{d\lambda}$ is the curve $\vec{U}$. If we parallel transport $\vec{A}\cdot\vec{B}$ along the curve, then:

$$\begin{aligned}\frac{d}{d\lambda}\left(g_{\alpha\beta}A^\alpha B^\beta\right) = \frac{\partial}{\partial x^\beta}\left(g_{\alpha\beta}A^\alpha B^\alpha\right)\frac{dx^\beta}{d\lambda}\\ = \left(A^\alpha B^\alpha\frac{\partial g_{\alpha\beta}}{\partial x^\beta} + g_{\alpha\beta}B^\alpha\frac{\partial V^\alpha}{\partial x^\beta} + g_{\alpha\beta}A^\alpha\frac{\partial B^\alpha}{\partial x^\beta}\right)\frac{dx^\beta}{d\lambda}\end{aligned} \tag{104}$$

Note that parallel transport requires the second and third term of this expression to equal zero and that the local flatness theorem ($g_{\alpha\beta,\gamma}=0$) requires the first term to equal zero, leaving us to conclude that $\vec{A}\cdot\vec{B}$ is constant along the curve $\vec{U}$.

**Exercise (6.13b):** Conclude from this that if a *geodesic* is spacelike (or timelike or null) somewhere, it is spacelike (or timelike or null) everywhere.

**Solution:** The conditions for determining whether a geodesic $\vec{U}|$ is spacelike, timelike, or null are given below:

$$\vec{U}\cdot\vec{U} = \begin{cases} <0 & \text{timelike}\\ =0 & \text{null}\\ >0 & \text{spacelike}\end{cases} \tag{105}$$

Since we have shown that $\vec{U}\cdot\vec{U}$ is constant, we know that if the geodesic is defined to be spacelike, timelike, or null anywhere, it must satisfy this condition everywhere.

**Exercise (6.14):** The proper distance along a curve whose tangent is $\vec{V}$ is given by Eq. (6.8). Show that if the curve is a geodesic, then the proper length is an affine parameter. (Use the result of Exer. 13.)

**Solution:** From Eq. (6.8), the proper distance is defined to be:

$$\ell = \int_{\lambda_0}^{\lambda_1}|\vec{V}\cdot\vec{V}|^{1/2}d\lambda \tag{106}$$

And since we know that $\vec{V}\cdot\vec{V}$ is a constant from Exer. (6.13), this integral will just result in $\vec{V}\cdot\vec{V}$ being multiplied by the length of the line:

$$\ell = |\vec{V}\cdot\vec{V}|^{1/2}\int_{\lambda_0}^{\lambda_1}d\lambda = |\vec{V}\cdot\vec{V}|^{1/2}\lambda \tag{107}$$

And since an affine parameter is defined to be $\phi = a\lambda + b$ on page 167 of the text, $\ell$ must be an affine parameter with $a = |\vec{V}\cdot\vec{V}|^{1/2}$ and $b=0$ (since, again, $\vec{V}\cdot\vec{V}$ was found to be a constant from the previous exercise).

**Exercise 6.19:** Prove that $R^\alpha_{\beta\mu\nu}=0$ for polar coordinates in the Euclidean plane. Use Eq. (5.44) or equivalent results.

**Solution:** Using the definition of the Riemann curvature tensor given by Eq. (6.63):

$$R^\alpha_{\beta\mu\nu} = \frac{\partial}{\partial x^\mu}\Gamma_{\beta\nu} - \frac{\partial}{\partial x^\nu}\Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu} \tag{108}$$

Where the $\alpha,\beta,\mu,\nu,\sigma$ indices will be summed over $r,\theta$ in polar coordinates. It's best to consider these sums in parts since they will become so large. Starting with the first term in the expression for $R^\alpha_{\beta\mu\nu}$:

$$\frac{\partial}{\partial x^\mu}\Gamma_{\beta\nu} = \frac{\partial}{\partial x^\mu}\left(\Gamma^\alpha_{rr}+\Gamma^\alpha_{r\theta}+\Gamma^\alpha_{\theta r}+\Gamma^\alpha_{\theta\theta}\right) = \frac{\partial}{\partial x^\mu}\left(\Gamma^r_\theta+\Gamma^\theta_{r\theta}+\Gamma^\theta_{\theta r}\right) = \frac{-2}{r^2}-r^2 \tag{109}$$

And since the $-\frac{\partial}{\partial x^\nu}\Gamma^\alpha_{\beta\mu}$ component of the Riemann curvature tensor changes nothing but the sign of the result shown above, $\frac{\partial}{\partial x^\mu}\Gamma_{\beta\nu} - \frac{\partial}{\partial x^\nu}\Gamma^\alpha_{\beta\mu}=0$. Computing the $\Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu}$ component of $R^\alpha_{\beta\mu\nu}$:

$$\Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} = \Gamma^r_{\theta\theta}\Gamma^\theta_{\beta\nu} = \Gamma^r_{\theta\theta}\Gamma^\theta_{r\theta}+\Gamma^r_{\theta\theta}\Gamma^\theta_{\theta r}+\Gamma^\theta_{r\theta}\Gamma^r_{\theta\theta} = -3 \tag{110}$$

And, again, since $-\Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu}$ changes nothing but the sign of the previous result, $\Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu}=0$, resulting in $R^\alpha_{\beta\mu\nu}=0$, which should be expected since we are computing the Riemann curvature tensor in the Euclidean plane, which is defined to have no curvature.

**Exercise (6.20):** Fill in the algebra necessary to establish Eq. (6.73).

**Solution:** Starting from:

$$\nabla_\alpha\nabla_\beta V^\mu \tag{111}$$

And using the definition of a covariant derivative built in the previous chapter, specifically Eq. (5.48):

$$\nabla_\alpha \nabla_\beta V^\mu = \nabla_\alpha \left( \frac{\partial V^\mu}{\partial x^\beta} + V^\nu \Gamma^\mu_{\nu\beta} \right) \tag{112}$$

The covariant derivative with respect to $\alpha$ can then be distributed across this expression like so:

$$\nabla_\alpha \left( \frac{\partial V^\mu}{\partial x^\beta} + V^\nu \Gamma^\mu_{\nu\beta} \right) = \frac{\partial V^\mu}{\partial x^\alpha x^\beta} + V^\nu \frac{\partial \Gamma^\mu_{\nu\beta}}{\partial x^\alpha} + \Gamma^\mu_{\nu\beta} \frac{\partial V^\nu}{\partial x^\alpha} \tag{113}$$

And since we are considering these covariant derivatives in a locally inertial frame at some point, the $\Gamma^\mu_{\nu\beta}$ term goes to zero but its partial derivative does not, leaving us with:

$$\nabla_\alpha \nabla_\beta V^\mu = \frac{\partial V^\mu}{\partial x^\alpha x^\beta} + V^\nu \frac{\partial \Gamma^\mu_{\nu\beta}}{\partial x^\alpha} \tag{114}$$

**Exercise (6.28a):** Derive Eq. (6.19) by using the usual coordinate transformation from Cartesian to spherical polars.

**Solution:** Using the transformation rule:

$$\vec{e}_{\beta'} = \Lambda^{\beta'}_{\alpha} \vec{e}_\beta \tag{115}$$

and the coordinates:

$$x = r \sin\theta \cos\phi \tag{116}$$
$$y = r \sin\theta \sin\phi \tag{117}$$
$$z = r \cos\theta \tag{118}$$

this implies that $\vec{e}_r$ will be of the form:

$$\vec{e}_r = \frac{\partial x}{\partial r}\vec{e}_x + \frac{\partial y}{\partial r}\vec{e}_y + \frac{\partial z}{\partial r}\vec{e}_z \tag{119}$$
$$\implies \vec{e}_r = \sin\theta \cos\phi \vec{e}_x + \sin\theta \sin\phi \vec{e}_y + \cos\theta \vec{e}_z \tag{120}$$

and that $\vec{e}_\theta$ and $\vec{e}_\phi$ will be of the forms:

$$\vec{e}_\theta = \frac{\partial x}{\partial r\theta}\vec{e}_x + \frac{\partial y}{\partial \theta}\vec{e}_y + \frac{\partial z}{\partial \theta}\vec{e}_z \tag{121}$$
$$\implies \vec{e}_\theta = r\cos\theta \vec{e}_x + r\cos\theta \sin\phi \vec{e}_y - r\sin\theta \vec{e}_z \tag{122}$$

$$\vec{e}_\phi = -r\sin\theta \sin\phi \vec{e}_x + r\sin\theta\cos\phi \vec{e}_y \tag{123}$$

From these, the following terms can be computed:

$$\vec{e}_r \cdot \vec{e}_r = 1 \tag{124}$$
$$\vec{e}_\theta \cdot \vec{e}_\theta = r^2 \tag{125}$$
$$\vec{e}_\phi \cdot \vec{e}_\phi = r^2 \sin^2\theta \tag{126}$$
$$\tag{127}$$

and that $\vec{e}_r \cdot \vec{e}_\theta = \vec{e}_\theta \cdot \vec{e}_r = \vec{e}_r \cdot \vec{e}_\phi = \vec{e}_\phi \cdot \vec{e}_r = \vec{e}_\theta \cdot \vec{e}_\phi = \vec{e}_\phi \cdot \vec{e}_\theta = 0$. These terms give the metric in spherical coordinates the form:

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{128}$$

**Exercise (6.28b):** Deduce from Eq. (6.19) that the metric of the surface of a sphere of radius $r$ has components $g_{rr} = r^2, g_{\phi\phi} = r^2\sin^2\theta, g_{\theta\phi} = 0$ in the usual spherical coordinates.

**Solution:** It's clear from the metric shown in the previous expression that $g_{rr} = r^2, g_{\phi\phi} = r^2\sin^2\theta$, and $g_{\theta\phi} = 0$.

**Exercise (6.28c):** Find the components $g^{\alpha\beta}$ for the sphere.

**Solution:** Since the metric for spherical coordinates is diagonal, its inverse will just invert the non-zero components like so:

$$g^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{pmatrix} \tag{129}$$

**Exercise (6.33a):** A three-sphere is the three-dimensional surface in four-dimensional Euclidean space (coordinates $x, y, z, w$) given by the equation $x^2 + y^2 + z^2 + w^2 = r^2$, where $r$ is the radius of the sphere. Define new coordinates $(r, \theta, \phi, \chi)$ by the equations $w = r\cos\theta$, $z = r\sin\chi\cos\theta$, $x = r\sin\chi\sin\theta\cos\phi$, and $y = r\sin\chi\sin\theta\sin\phi$. Show that $(\theta, \phi, \chi)$ are coordinates for the sphere. These generalize the familiar polar coordinates.

**Solution:** To show that the $(\theta, \phi, \chi)$ coordinates define a four-dimensional sphere, we must compute $x^2 + y^2 + z^2 + w^2$. If this result results in the radius of the sphere, $r$, then we have defined coordinates that describe it. This computation is as follows:

$$x^2 + y^2 + z^2 + w^2 = r^2\sin^2\chi\sin^2\theta\cos^2\phi + r^2\sin^2\chi\sin^2\theta\sin^2\phi + r^2\sin^2\chi\cos^2\theta + r^2\cos^2\chi \tag{130}$$

Using the trigonometric identity $\sin^2\eta + \cos^2\eta = 1$ many times reduces this expression to the desired result that $x^2 + y^2 + z^2 + w^2 = r$, which means that our $(\theta, \phi, \chi)$ coordinates do in fact define a four-dimensional sphere.

**Exercise (6.33b):** Show that the metric of the three-sphere of radius $r$ has components in these coordinates $g_{\chi\chi} = r^2, g_{\theta\theta} = r^2\sin^2\chi, g_{\phi\phi} = r^2\sin^2\chi\sin^2\theta$, all other components vanishing. (Use the same method as in Exer. 28.)

**Solution:** Using the same method as in Exer. 28, it's found that:

$$\vec{e}_r = \sin\chi\sin\theta\cos\phi\vec{e}_x + \sin\chi\sin\theta\sin\phi\vec{e}_y + \sin\chi\cos\theta\vec{e}_z + \cos\chi\vec{e}_w \tag{131}$$

$$\vec{e}_\theta = r\sin\chi\cos\theta\cos\phi\vec{e}_x + r\sin\chi\cos\theta\sin\phi\vec{e}_y - r\sin\chi\sin\theta\vec{e}_z \tag{132}$$

$$\vec{e}_\phi = -r\sin\chi\sin\theta\vec{e}_x + r\sin\chi\sin\theta\cos\phi\vec{e}_y \tag{133}$$

$$\vec{e}_\chi = r\cos\chi\sin\theta\cos\phi\vec{e}_x + r\cos\chi\sin\theta\sin\phi\vec{e}_y + r\cos\chi\cos\theta\vec{e}_z - r\sin\chi\vec{e}_w \tag{134}$$

Dotting all of these terms with each other results in the metric:

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r^2\sin^2\theta & 0 & 0 \\ 0 & 0 & r^2\sin^2\chi\sin^2\theta & 0 \\ 0 & 0 & 0 & r^2 \end{pmatrix} \tag{135}$$

which is the desired result.

# 7 Chapter 7: Physics in a Curved Spacetime

**Exercise (7.2):** To first order in $\phi$, compute $g^{\alpha\beta}$ for Eq. (7.8).

**Solution:** Eq. (7.8) gives the line element for the ordinary Newtonian potential to the first order to be:

$$ds^2 = -(1 + 2\phi)dt^2 + (1 - 2\phi)(dx^2 + dy^2 + dz^2) \tag{136}$$

so the metric can be inferred to be:

$$g_{\alpha\beta} = \begin{pmatrix} -(1+2\phi) & 0 & 0 & 0 \\ 0 & (1-2\phi) & 0 & 0 \\ 0 & 0 & (1-2\phi) & 0 \\ 0 & 0 & 0 & (1-2\phi) \end{pmatrix} \tag{137}$$

Since this metric is diagonal, the inverse of it will just invert the components, meaning that:

$$g^{\alpha\beta} = \begin{pmatrix} \frac{-1}{1+2\phi} & 0 & 0 & 0 \\ 0 & \frac{1}{1-2\phi} & 0 & 0 \\ 0 & 0 & \frac{1}{1-2\phi} & 0 \\ 0 & 0 & 0 & \frac{1}{1-2\phi} \end{pmatrix} \tag{138}$$

**Exercise (7.3):** Calculate all the Christoffel symbols for the metric given by Eq. (7.8), to first order in $\phi$. Assume $\phi$ is a general function of $t, x, y$, and $z$.

**Solution:** Using Eq. (5.75), which allows us to compute the Christoffel symbols in terms of the metric and its derivatives, the components of the Christoffel symbols can be computed. Take $\Gamma^x_{xx}$, for example. This would be computed by:

$$\begin{aligned}\Gamma^x_{xx} &= \frac{1}{2}g^{tx}\left(g_{tx,x} + g_{tx,x} - g_{xx,t}\right) + \frac{1}{2}g^{xx}\left(g_{xx,x} + g_{xx,x} - g_{xx,x}\right)\\ &+\frac{1}{2}g^{yx}\left(g_{yx,x} + g_{yx,x} - g_{xx,y}\right) + \frac{1}{2}g^{zx}\left(g_{zx,x} + g_{zx,x} - g_{xx,z}\right)\end{aligned} \tag{139}$$

Which reduces nicely since the non-zero components of this metric are only found along the diagonal, meaning that only $g^{tt}, g^{xx}, g^{yy}, g^{zz}$ will have non-zero components. This results in:

$$\begin{aligned}\Gamma^x_{xx} &= \frac{1}{2}g^{xx}\left(g_{xx,x} + g_{xx,x} - g_{xx,x}\right) = \frac{1}{2}g^{xx}\frac{\partial g_{xx}}{\partial x}\\ &= \frac{1}{2}(1-2\phi)\frac{\partial}{\partial x}(1-2\phi) = -(1-2\phi)\frac{\partial \phi}{\partial x}\end{aligned} \tag{140}$$

And the other 63 components of $\Gamma^\gamma_{\beta\mu}$ can be computed in the same way, using the simplifications used above.

# 8 Chapter 8: The Einstein Field Equations

**Exercise (8.1):** Show that Eq. (8.2) is a solution of Eq. (8.1) by the following method. Assume the point particle to be at the origin, $r = 0$, and to produce a spherically symmetric field. Then use Gauss' law on a sphere of radius $r$ to conclude

$$\frac{d\phi}{dr} = \frac{Gm}{r^2}$$

Deduce Eq. (8.2) from this. (consider the behavior at infinity.)

**Solution:** First considering the acceleration experienced due to the gravitational field around this point particle, one gets:

$$g = \frac{Gm}{r^2} \tag{141}$$

When considering the application of Gauss' law to this object, it's clear that the direction of $g$ will be opposite to that of $dA$, meaning $\vec{g}\cdot d\vec{A} = -gdA$. Using this in the integral form of Gauss' law when the integrated area is the surface area of a sphere concentric around the point at the origin:

$$\phi = \iint \vec{g}\cdot d\vec{A} = -\iint gdA = g - 4\pi r^2 = -4\pi Gm \tag{142}$$

Since the integral form of Gauss' law must be equal to the differential form:

$$\iiint \nabla \cdot gdV = -4\pi Gm \tag{143}$$

Since mass is just the density of some object integrated over a volume:

$$\iiint \nabla\cdot gdV = -4\pi G\iiint \rho dV \tag{144}$$

Which means by inspection that:

$$\nabla \cdot g = -4\pi G\rho \tag{145}$$

Given the form of the scalar gravitational potential $\phi = Gm/r$ and Eq. (141), it's clear that

$$g = -\frac{d\phi}{dr} \tag{146}$$

And since $\phi$ is only a function of $r$, $-d\phi/dr \equiv -\nabla\phi$ which means we can simplify Eq. (145) to the desired form:

$$\nabla^2\phi = 4\pi G\rho \tag{147}$$

**Exercise (8.5a):** Show that if $h_{\alpha\beta} = \xi_{\alpha,\beta} + \xi_{\beta,\alpha}$, then Eq. (8.25) vanishes.

**Solution:** Starting with Eq. (8.25):

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu}\right) \tag{148}$$

All for components of $R_{\alpha\beta\mu\nu}$ must be computed in the following way to show that it vanishes:

$$h_{\alpha\nu,\beta\mu} = \xi_{\alpha,\beta\mu\nu} + \xi_{\nu,\alpha\beta\mu} \tag{149}$$

$$h_{\beta\mu,\alpha\nu} = \xi_{\beta,\alpha\mu\nu} + \xi_{\mu,\alpha\beta\nu} \tag{150}$$

$$-h_{\alpha\mu,\beta\nu} = -\xi_{\alpha,\beta\mu\nu} - \xi_{\mu,\alpha\beta\nu} \tag{151}$$

$$-h_{\beta\nu,\alpha\mu} = -\xi_{\beta,\alpha,\mu\nu} - \xi_{\nu,\alpha\beta\mu} \tag{152}$$

It's apparent that when you add Eq. (115) - Eq. (118) together, you will get 0, meaning $R_{\alpha\beta\mu\nu} = 0$.

**Exercise (8.5b):** Argue from this that Eq. (8.25) is gauge invariant.

**Solution:** A gauge transformation is defined as a small change in coordinates where $h_{\alpha\beta} \to h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$. Since we have just shown that $h_{\alpha\beta} = \xi_{\alpha,\beta} + \xi_{\beta,\alpha}$ results in $R_{\alpha\beta\mu\nu} = 0$, if we use this expression for $h_{\alpha\beta}$ in the gauge transformation, we get $h_{\alpha\beta} \to \xi_{\alpha,\beta} + \xi_{\beta,\alpha} + \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$, which means that the gauge transformation becomes $h_{\alpha\beta} \to 0$. Using this expression for the transformed coordinates in Eq. (8.25) will obviously result in $R_{\alpha\beta\mu\nu} = 0$, which means that $R_{\alpha\beta\mu\nu}$ has been unchanged under this gauge transformation, meaning it is gauge invariant.

**Exercise (8.6):** Weak-field theory assumes $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$. Similarly $g^{\mu\nu}$ must be close to $\eta^{\mu\nu}$, say $g^{\mu\nu} = \eta^{\mu\nu} + \delta g^{\mu\nu}$. Show from Exer. 4a that $\delta g^{\mu\nu} = -h^{\mu\nu} + O(h^2)$. Thus, $\eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$ is *not* the deviation of $g^{\mu\nu}$ from flatness.

**Solution:** Start with the assumption that:

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta h_{\mu\nu} \tag{153}$$

and

$$g^{\mu\sigma} = \eta^{\mu\sigma} + \delta g^{\mu\sigma} \tag{154}$$

then:

$$g_{\mu\nu}g^{\mu\sigma} = \left(\eta_{\mu\nu} + \delta h_{\mu\nu}\right)\left(\eta^{\mu\sigma} + \delta g^{\mu\sigma}\right) \tag{155}$$

It is an identity that $g_{\alpha\beta}g^{\beta\lambda} = \delta^{\lambda}_{\alpha}$ so, after multiplying the terms out, this expression becomes:

$$\delta^{\sigma}_{\mu} = \eta_{\mu\nu}\eta^{\mu\sigma} + \eta_{\mu\nu}\delta g^{\mu\sigma} + h_{\mu\nu}\eta^{\nu\sigma} + h_{\mu\nu}\delta g^{\nu\sigma} \tag{156}$$

We can use the identity $\eta_{\alpha\beta}\eta^{\beta\lambda} = \delta^{\lambda}_{\alpha}$ again to get:

$$\delta^{\sigma}_{\mu} = \delta^{\sigma}_{\mu} + \eta_{\mu\nu}\delta g^{\mu\sigma} + h_{\mu\nu}\eta^{\nu\sigma} + h_{\mu\nu}\delta g^{\nu\sigma} \tag{157}$$

So the $\delta^{\sigma}_{\mu}$ terms cancel out, leaving us with:

$$0 = \eta_{\mu\nu}\delta g^{\mu\sigma} + h_{\mu\nu}\eta^{\nu\sigma} + h_{\mu\nu}\delta g^{\nu\sigma} \tag{158}$$

Since we have assumed that $|h_{\mu\nu}| \ll 1$ and also expect that $|\delta g^{\mu\sigma}| \ll 1$, we should expect that their product, $h_{\mu\nu}\delta g^{\mu\sigma}$ should be nearly zero. This assumption allows us to drop this term in the previous expression so we get:

$$-\eta_{\mu\nu}\delta g^{\mu\sigma} = h_{\mu\nu}\eta^{\nu\sigma} \tag{159}$$

Solving for $\delta g^{\mu\sigma}$ results in:

$$\delta g^{\mu\sigma} = -\eta^{\nu\sigma}\eta^{\mu\nu}h_{\mu\nu} \tag{160}$$

These two Minkowski metrics act to swap and raise the indices on $h_{\mu\nu}$ so this gives us:

$$\delta g^{\mu\sigma} = -h^{\nu\sigma} \tag{161}$$

Which is the desired result. Notice that the exercise includes a $O(h^2)$ factor that would have remained in this expression if I had not just taken $h_{\mu\nu}\delta g^{\mu\sigma}$ out of the expression entirely.

# 9 Chapter 9: Gravitational Radiation

**Exercise (9.2):** Show that the real and imaginary parts of Eq. (9.2) at a fixed spatial position $\{x^i\}$ oscillate sinusoidally in time with frequency $\omega = k^0$.

**Solution:** Starting with Eq. (9.2):

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta} e^{ik_\alpha x^\alpha} \tag{162}$$

With the use of Euler's formula, this becomes:

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta} \left(\cos(k_\alpha x^\alpha) + i\sin(k_\alpha x^\alpha)\right) \tag{163}$$

Isolating the $\alpha = 0$ component in this expression leaves us with:

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta} \left(\cos(k_0 x^0) + i\sin(k_0 x^0) + \cos(k_i x^i) + i\sin(k_i x^i)\right) \tag{164}$$

Which shows that $\bar{h}^{\alpha\beta}$ oscillates as a sinusoid in time with an angular frequency $\omega = k_0$:

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta} \left(\cos(\omega t) + i\sin(\omega t) + \cos(k_i x^i) + i\sin(k_i x^i)\right) \tag{165}$$

# 10 Chapter 10: Spherical Solutions for Stars

**Exercise (10.1):** Starting with $ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$, show that the coordinate transformation $r = \sqrt{x^2 + y^2 + z^2}$, $\theta = \cos^{-1}(z/r)$, $\phi = \tan^{-1}(y/x)$ leads to Eq. (10.1), $ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$.

**Solution:** To derive this line element, we must first compute the metric for a flat spacetime in spherical coordinates. This metric will follow the form:

$$g_{\alpha\beta} = \begin{pmatrix} \vec{e}_t \cdot \vec{e}_t & \vec{e}_t \cdot \vec{e}_r & \vec{e}_t \cdot \vec{e}_\theta & \vec{e}_t \cdot \vec{e}_\phi \\ \vec{e}_r \cdot \vec{e}_t & \vec{e}_r \cdot \vec{e}_r & \vec{e}_r \cdot \vec{e}_\theta & \vec{e}_r \cdot \vec{e}_\phi \\ \vec{e}_\theta \cdot \vec{e}_t & \vec{e}_\theta \cdot \vec{e}_r & \vec{e}_\theta \cdot \vec{e}_\theta & \vec{e}_\theta \cdot \vec{e}_\phi \\ \vec{e}_\phi \cdot \vec{e}_t & \vec{e}_\phi \cdot \vec{e}_r & \vec{e}_\phi \cdot \vec{e}_\theta & \vec{e}_\phi \cdot \vec{e}_\phi \end{pmatrix} \tag{166}$$

Since this transformation from Cartesian to spherical coordinates does not depend on $t$, $\vec{e}_t$ will be of the form $\langle dt, 0, 0, 0\rangle$ and $\vec{e}_i$ will be of the form $\langle 0, ..., ..., ...\rangle$, which means that $\vec{e}_t \cdot \vec{e}_i = 0$. This turns our metric into:

$$g_{\alpha\beta} = \begin{pmatrix} \vec{e}_t \cdot \vec{e}_t & 0 & 0 & 0 \\ 0 & \vec{e}_r \cdot \vec{e}_r & \vec{e}_r \cdot \vec{e}_\theta & \vec{e}_r \cdot \vec{e}_\phi \\ 0 & \vec{e}_\theta \cdot \vec{e}_r & \vec{e}_\theta \cdot \vec{e}_\theta & \vec{e}_\theta \cdot \vec{e}_\phi \\ 0 & \vec{e}_\phi \cdot \vec{e}_r & \vec{e}_\phi \cdot \vec{e}_\theta & \vec{e}_\phi \cdot \vec{e}_\phi \end{pmatrix} \tag{167}$$

To compute $\vec{e}_r$, $\vec{e}_\theta$, and $\vec{e}_\phi$, the transformation $\vec{e}_{\alpha'} = \Lambda^\beta_{\alpha'} \vec{e}_\beta$ will be used with $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$:

$$\begin{gathered} \vec{e}_r = \frac{\partial x}{\partial r}\vec{e}_x + \frac{\partial y}{\partial r}\vec{e}_y + \frac{\partial z}{\partial r}\vec{e}_z \\ \implies \vec{e}_r = \sin\theta\cos\phi\vec{e}_x + \sin\theta\sin\phi\vec{e}_y + \cos\theta\vec{e}_z \end{gathered} \tag{168}$$

$$\begin{gathered} \vec{e}_\theta = \frac{\partial x}{\partial \theta}\vec{e}_x + \frac{\partial y}{\partial \theta}\vec{e}_y + \frac{\partial z}{\partial \theta}\vec{e}_z \\ \implies \vec{e}_\theta = r\cos\theta\cos\phi\vec{e}_x + r\cos\theta\sin\phi\vec{e}_y - r\sin\theta\vec{e}_z \end{gathered} \tag{169}$$

$$\begin{gathered} \vec{e}_\phi = \frac{\partial x}{\partial \phi}\vec{e}_x + \frac{\partial y}{\partial \phi}\vec{e}_y + \frac{\partial z}{\partial \phi}\vec{e}_z \\ \implies \vec{e}_\phi = -r\sin\theta\sin\phi\vec{e}_x + r\sin\theta\cos\phi\vec{e}_y \end{gathered} \tag{170}$$

By direct computation, it can then be shown that $\vec{e}_\theta \cdot \vec{e}_r = \vec{e}_r \cdot \vec{e}_\theta = \vec{e}_r \cdot \vec{e}_\phi = \vec{e}_\phi \cdot \vec{e}_r = \vec{e}_\theta \cdot \vec{e}_\phi = \vec{e}_\phi \cdot \vec{e}_\theta = 0$, which are notably all of the off-diagonal elements in $g_{\alpha\beta}$. It can also then be shown by direct computation that $\vec{e}_r \cdot \vec{e}_r = 1$, $\vec{e}_\theta \cdot \vec{e}_\theta = r^2$, and $\vec{e}_\phi \cdot \vec{e}_\phi = r^2\sin^2\theta$. This results in the metric:

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{171}$$

Reading the line element $ds^2$ off of this metric results in:

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{172}$$

# 11 Chapter 12: Cosmology

**Exercise (12.7a):** Find the coordinate transformation leading to Eq. (12.20).

**Solution:** To consider the Robertson-Walker metric:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega\right) \tag{173}$$

and when $k = -1$ requires the coordinate transformation:

$$d\chi^2 = \frac{dr^2}{1+r^2} \tag{174}$$

This implies that

$$d\chi = \frac{dr}{\sqrt{1+r^2}} \tag{175}$$

Integrating this to get $\chi(r)$ results in

$$\chi = \sinh^{-1}(r) \implies r = \sinh(\chi) \tag{176}$$

With this transformation, the Robertson-Walker metric when $k = -1$ becomes:

$$ds^2 = -dt^2 + a^2(t)\left(d\chi^2 + \sinh^2(\chi) d\Omega\right) \tag{177}$$