A First Course in Loop Quantum Gravity - Selected Solutions

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January 2024 - Present

Chapter 2: Special Relativity and Electromagnetism

Problem (2.3): Show explicitly the boost (2.11) indeed leaves the distance (2.8) invariant.

Solution: The distance, (2.8), is given as:

$$ds^2 = -(cdt)^2 + dx^2 + dy^2 + dz^2 \tag{1}$$

which I will write as:

$$(dx^\mu)^2 = \begin{pmatrix} -dt^2 \\ dx^2 \\ dy^2 \\ dz^2 \end{pmatrix} \tag{2}$$

where $(dx^0)^2 = -dt^2$ and $c = 1$ for simplicity.

The boost (2.11) is given as:

$$\Lambda^{\mu'}_{\mu} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

where ϕ is a parameter, ranging from $-\infty$ to ∞, given by $\phi = \tanh^{-1}(v)$.

To transform this interval, it can be multiplied by Eq. (3):

$$(dx^{\mu'})^2 = \Lambda^{\mu'}_{\mu}(dx^\mu)^2 = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -dt^2 \\ dx^2 \\ dy^2 \\ dz^2 \end{pmatrix} \tag{4}$$

This results in the matrix:

$$(dx^{\mu'})^2 = \begin{pmatrix} \cosh\phi dt^2 - \sinh\phi dx^2 \\ -\sinh\phi dt^2 + \cosh\phi dx^2 \\ dy^2 \\ dz^2 \end{pmatrix} \tag{5}$$

Recalling that $\phi = \tan^{-1}(v)$, $(dx^{\mu'})^2$ can be simplified down to:

$$(dx^{\mu'})^2 = \begin{pmatrix} \gamma^2(dt^2 - v^2dx^2) \\ \gamma^2(dx^2 - v^2dt^2) \\ dy^2 \\ dz^2 \end{pmatrix} \tag{6}$$

where $\gamma = 1/\sqrt{1-v^2}$.

So to show that the spactime interval is invariant under the Lorentz transformation, we need to show that $-(dt')^2 + (dx')^2 = -dt^2 + dx^2$:

$$\begin{aligned} -(dt')^2 + (dx')^2 &= \gamma^2 dt^2 - v^2\gamma^2 dx^2 + \gamma^2 dx^2 - v^2\gamma^2 dt^2 \\ &= \gamma^2\left(1 - v^2\right)dt^2 + \gamma^2\left(1 - v^2\right)dx^2 \\ &= dt^2 + dx^2 \\ \implies -(dt')^2 + (dx')^2 &= dt^2 + dx^2 \end{aligned} \tag{7}$$

And since $(dy')^2 = dy^2$ and $(dz')^2 = dz^2$, we have shown that the spacetime interval is invariant under Lorentz transformations: $(ds')^2 = ds^2$

Problem (2.4): Show that $\epsilon^{\sigma\alpha\beta\gamma}\partial_\gamma F_{\alpha\beta} = 0$ is equivalent to half of the Maxwell equations.

Solution: With the knowledge that the Levi-Civita symbol, $\epsilon^{\sigma\alpha\beta\gamma}$, is equal to one only when there is an even permutation in the $\sigma\alpha\beta\gamma$ indices, this becomes:

$$\epsilon^{\sigma\alpha\beta\gamma} = \begin{cases} 1 & \text{if } \sigma\alpha\beta\gamma \text{ is an even permutation} \\ 0 & \text{repeated indices} \\ -1 & \text{else} \end{cases} \tag{8}$$

in four dimensions where $0 \to t$, $1 \to x$, $2 \to y$, and $3 \to z$. We can begin to arrive half of the Maxwell equations by plugging values into the σ index of $\epsilon^{\sigma\alpha\beta\gamma}$ since this restrict the values of the other indices that do not result in zero. Starting with $\sigma = 0$:

$$\begin{aligned} \epsilon^{0\alpha\beta\gamma}\partial_\gamma F_{\alpha\beta} = 0 \\ \implies \epsilon^{0123}\partial_3 F_{12} + \epsilon^{0231}\partial_1 F_{23} + \epsilon^{0312}\partial_2 F_{31} + \epsilon^{0321}\partial_1 F_{32} + \epsilon^{0213}\partial_3 F_{21} + \epsilon^{0132}\partial_2 F_{13} = 0 \\ \implies \partial_3 F_{12} + \partial_1 F_{23} + \partial_2 F_{31} - \partial_1 F_{32} - \partial_3 F_{21} - \partial_2 F_{13} = 0 \implies \end{aligned} \tag{9}$$

And since $F_{\alpha\beta}$ is defined as the field equation below:

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{10}$$

Eq. (9) becomes:

$$\begin{aligned} 2\partial_1 B_1 + 2\partial_2 B_2 + 2\partial_3 B_3 = 0 \\ \implies \partial_x B_x + \partial_y B_y + \partial_z B_z = 0 \\ \implies \nabla \cdot \vec{B} = 0 \end{aligned} \tag{11}$$

Now considering $\sigma = 1$, this results in:

$$\begin{aligned} \epsilon^{1\alpha\beta\gamma}\partial_\gamma F^{\alpha\beta} = \partial_t B_x + \partial_y E_z - \partial_z E_y + \partial_y E_z + \partial_t B_x - \partial_z E_y = 0 \\ \implies 2\left(\partial_t B_x + \partial_y E_z - \partial_z E_y\right) \implies \partial_t B_x + \partial_y E_z - \partial_z E_y = 0 \end{aligned} \tag{12}$$

$\sigma = 2$ results in:

$$\begin{aligned} \partial_x E_z - \partial_z E_x - \partial_t B_y + \partial_x E_z - \partial_t B_y - \partial_z E_x = 0 \\ \implies -2\left(\partial_t B_y - \partial_x E_z + \partial_z E_x\right) = 0 \implies \partial_t B_y - \partial_x E_z + \partial_z E_x = 0 \end{aligned} \tag{13}$$

$\sigma = 3$ results in:

$$\begin{aligned} -\partial_y E_x + \partial_t B_z + \partial_x E_y - \partial_y E_x + \partial_x E_y + \partial_t B_z = 0 \\ \implies -\partial_y E_x + \partial_t B_z + \partial_x E_y = 0 \end{aligned} \tag{14}$$

Combining the $\sigma = 1, 2, 3$ results gives us:

$$\partial_t B_x + \partial_y E_z - \partial_z E_y + \partial_t B_y - \partial_x E_z + \partial_z E_x - \partial_y E_x + \partial_t B_z + \partial_x E_y = 0 \tag{15}$$

Notice here that all of the ∂_t terms make up the components of $\partial_t \vec{B}$ and all of the ∂_i terms, where $i \in \{x, y, z\}$ make up the components of $\nabla \times \vec{E}$, giving us the third Maxwell equation:

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{16}$$

Chapter 3: Some Elements of General Relativity

Problem (3.1):

Problem (3.2): Given the formula for the covariant derivative of a vector (3.10) V^μ, assume that the covariant derivative of a vector with its index down is given by:

$$\nabla_\mu V_\nu = \partial_\mu V_\nu + \bar{\Gamma}^\lambda_{\mu\nu} V_\lambda.$$

Show that for the derivative of a scalar $V_\mu V^\mu$ to be given by just the partial derivative, one has to have that $\bar{\Gamma}^\lambda_{\mu\nu} = -\lambda^\lambda_{\mu\nu}$. Therefore the covariant derivative of a vector with its index down is given by:

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda_{\mu\nu} V_\lambda.$$

For the derivative of a scalar $V_\mu V^\mu$ to be given by just the partial derivative,

$$\nabla_\mu \left(V_\nu V^\nu\right) = \partial_\mu \left(V_\nu V^\nu\right) \tag{17}$$

must be satisfied. Since the partial derivative operator and covariant derivative operators are linear, it must be true that

$$V^\nu \nabla_\mu V_\nu + V^\nu \nabla_\mu V^\nu = V^\nu \partial_\mu V_\nu + V_\nu \partial_\mu V^\nu. \tag{18}$$

We know the definition of $\nabla_\mu V^\nu$ so this brings the expression to:

$$V^\nu \nabla_\mu V_\nu + V_\nu \left(\partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda\right) = V^\nu \partial_\mu V_\nu + V_\nu \partial_\mu V^\nu. \tag{19}$$

We can expand the $\nabla_\mu V_\nu$ component with our assumption of what form the covariant derivative of a covariant vector will be so this becomes:

$$V^\nu \left(\partial_\mu V_\nu + \bar{\Gamma}^\lambda_{\mu\nu} V_\lambda\right) + V_\nu \left(\partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda\right) = V^\nu \partial_\mu V_\nu + V_\nu \partial_\mu V^\nu. \tag{20}$$

It's clear that the partial derivative terms will cancel, leaving us with:

$$V^\nu \bar{\Gamma}^\lambda_{\mu\nu} V_\lambda = -V_\nu \Gamma^\nu_{\mu\lambda} V^\lambda. \tag{21}$$

It's crucial to notice that both sides of this expression, when summed over all relevant indices, will only be in terms of μ. Because of this, ν and λ are dummy indices. Making the index swap so that $\nu \to \lambda$ and $\lambda \to \nu$ on the right hand side of this expression results in:

$$V^\nu \bar{\Gamma}^\lambda_{\mu\nu} V_\lambda = -V_\lambda \Gamma^\lambda_{\mu\nu} V^\nu \tag{22}$$

This allows us to swap the positions of the V^ν and V_λ terms on the right hand side of this expression, leaving us with:

$$V^\nu \bar{\Gamma}^\lambda_{\mu\nu} V_\lambda = -V^\nu \Gamma^\lambda_{\mu\nu} V_\lambda \tag{23}$$

The V^ν and V_λ terms appear on both sides of the expression so can be cancelled to result in an expression only comparing $\Gamma^\lambda_{\mu\nu}$ and $\bar{\Gamma}^\lambda_{\mu\nu}$:

$$\bar{\Gamma}^\lambda_{\mu\nu} = -\Gamma^\lambda_{\mu\nu} \tag{24}$$

which is the desired result.

Problem (3.3): The general formula for the covariant derivative of a tensor with k upper indices and l lower indices is given by

$$\begin{aligned}\nabla_\lambda T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l} = \partial_\lambda T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l} + \Gamma^{\mu_1}_{\lambda\sigma} T^{\sigma\ldots\mu_k}_{\nu_1\ldots\nu_l} + \cdots + \Gamma^{\mu_1}_{\lambda\sigma} T^{\mu_1\ldots\sigma}_{\nu_1\ldots\nu_l} \\ -\Gamma^\sigma_{\lambda\nu_1} T^{\mu_1\ldots\mu_k}_{\sigma\ldots\nu_l} - \cdots - \Gamma^\sigma_{\lambda\nu_l} T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\sigma}.\end{aligned} \tag{25}$$

Show that the covariant derivative of the metric vanishes if the connection is the Christoffel connection.

Solution: Including only the relevant terms in the general form of the covariant derivative of any tensor when one considers a metric $g_{\mu\nu}$ leaves us with:

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\sigma_{\lambda\mu} g_{\sigma\nu} - \Gamma^\sigma_{\lambda\nu} g_{\mu\sigma}. \tag{26}$$

The definition of the Christoffel symbols alters this expression to be of the form:

$$\begin{aligned}\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \frac{1}{2} g_{\sigma\nu} g^{\sigma\rho} \left(\partial_\lambda g_{\rho\mu} + \partial_\mu g_{\rho\lambda} - \partial_\rho g_{\lambda\mu}\right) \\ -\frac{1}{2} g_{\mu\sigma} g^{\sigma\rho} \left(\partial_\lambda g_{\nu\rho} + \partial_\nu g_{\lambda\rho} - \partial_\rho g_{\lambda\nu}\right)\end{aligned} \tag{27}$$

The metrics being multiplied by each other all have σ terms so they become Kronecker delta functions:

$$\begin{aligned}\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \frac{1}{2} \delta^\rho_\nu \left(\partial_\lambda g_{\rho\mu} + \partial_\mu g_{\rho\lambda} - \partial_\rho g_{\lambda\mu}\right) \\ -\frac{1}{2} \delta^\rho_\mu \left(\partial_\lambda g_{\nu\rho} + \partial_\nu g_{\lambda\rho} - \partial_\rho g_{\lambda\nu}\right)\end{aligned} \tag{28}$$

which then make the change of variables $\rho \to \nu$ for the left-hand expression and $\mu \to \rho$ for the right-hand expression, leaving us with:

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \frac{1}{2}\left(\partial_\lambda g_{\nu\mu} + \partial_\mu g_{\nu\lambda} - \partial_\rho g_{\lambda\mu}\right) \\ -\frac{1}{2}\left(\partial_\lambda g_{\nu\mu} + \partial_\nu g_{\lambda\mu} - \partial_\mu g_{\lambda\nu}\right). \tag{29}$$

By inspection, all of the terms above will cancel with each other since we have assumed that the metric is symmetric ($g_{\mu\nu} = g_{\nu\mu}$) so this leaves us with the desired result:

$$\nabla_\lambda g_{\mu\nu} = 0. \tag{30}$$

This result should be expected since the metric of our manifold should not change as we parallel transport vectors along curves on it.

Problem (3.4): Show that if the connection is symmetric and the covariant derivative of the metric vanishes, the connection is given by the Christoffel symbols using the method in the text.

Solution: If the covariant derivative of the metric vanishes, then

$$\nabla_\sigma g_{\mu\nu} = \nabla_\mu g_{\sigma\nu} = \nabla_\nu g_{\sigma\mu} = 0. \tag{31}$$

So we can write out the terms of these three statements with the original definition of the covariant derivative and sum them in a way such that we're only let with an expression for a single Christoffel symbol. Beginning with $\nabla_\sigma g_{\mu\nu}$:

$$\nabla_\sigma g_{\mu\nu} = \partial_\sigma g_{\mu\nu} - \Gamma^\lambda_{\sigma\nu} g_{\lambda\mu} - \Gamma^\lambda_{\sigma\mu} g_{\lambda\nu} = 0 \tag{32}$$

$$\nabla_\mu g_{\sigma\nu} = \partial_\mu g_{\sigma\nu} - \Gamma^\lambda_{\mu\nu} g_{\lambda\sigma} - \Gamma^\lambda_{\mu\sigma} g_{\lambda\nu} = 0 \tag{33}$$

$$\nabla_\nu g_{\sigma\mu} = \partial_\nu g_{\sigma\mu} - \Gamma^\lambda_{\nu\mu} g_{\lambda\sigma} - \Gamma^\lambda_{\nu\sigma} g_{\lambda\mu} = 0 \tag{34}$$

Now with some trial and error along with some luck, one realizes that if Eq. (32) and Eq. (33) are subtracted from Eq. (34) when the assumption that the Christoffel symbols are symmetric $\left(\Gamma^k_{ij} = \Gamma^k_{ji}\right)$, there will remain an expression in terms of a single Christoffel symbol once the smoke clears:

$$-\partial_\sigma g_{\mu\nu} + \Gamma^\lambda_{\sigma\nu} g_{\lambda\mu} + \Gamma^\lambda_{\sigma\mu} g_{\lambda\nu} - \partial_\mu g_{\sigma\nu} + \Gamma^\lambda_{\mu\nu} g_{\lambda\sigma} + \Gamma^\lambda_{\mu\sigma} g_{\lambda\nu} + \partial_\nu g_{\sigma\mu} - \Gamma^\lambda_{\nu\mu} g_{\lambda\sigma} - \Gamma^\lambda_{\nu\sigma} g_{\lambda\mu} \tag{35}$$

$$-\partial_\sigma g_{\mu\nu} - \partial_\mu g_{\sigma\nu} - \partial_\nu g_{\sigma\mu} - 2\Gamma^\lambda_{\sigma\mu} g_{\lambda\nu} = 0 \tag{36}$$

$$\Gamma^\lambda_{\sigma\mu} = \frac{1}{2} g^{\lambda\nu}\left(\partial_\sigma g_{\mu\nu} + \partial_\mu g_{\sigma\nu} - \partial_\nu g_{\sigma\mu}\right) \tag{37}$$

Problem (3.6): Compute $\rho(t)$ for the dust filled FRW model with $a(t) = t^{2/3}$.

Solution: Since we want an expression for $\rho(t)$, we are looking for the T_{00} term in the Einstein field equations:

$$\frac{1}{8\pi G}\left(R_{00} - \frac{1}{2} g_{00} R\right) = T_{00} = \rho(t) \tag{38}$$

with the diagonal metric:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & t^{2/3} & 0 & 0 \\ 0 & 0 & t^{2/3} & 0 \\ 0 & 0 & 0 & t^{2/3} \end{pmatrix}. \tag{39}$$

I will begin by solving the Ricci tensor R_{00} using its definition in terms of the Christoffel symbols:

$$R_{00} = \partial_k \Gamma^k_{00} - \partial_0 \Gamma^k_{0k} + \Gamma^k_{00}\Gamma^m_{km} - \Gamma^k_{0m}\Gamma^m_{0k}. \tag{40}$$

Now I will take a look at the individual Christoffel symbols:

$$\Gamma^k_{00} = \frac{1}{2} g^{k\rho}\left(\cancel{\partial_0 g_{\rho 0}} + \cancel{\partial_0 g_{\rho 0}} - \cancel{\partial_\rho g_{00}}\right) = 0 \tag{41}$$

Since $\Gamma^k_{00} = 0$, this reduces the Ricci tensor down to:

$$R_{00} = -\partial_0 \Gamma^k_{0k} + -\Gamma^k_{0m}\Gamma^m_{0k}. \tag{42}$$

Continuing on:

$$-\Gamma^k_{0m} = -\frac{1}{2} g^{k\rho} \left(\partial_0 g_{\rho m} + \cancel{\partial_m g_{\rho 0}} - \cancel{\partial_\rho g_{0m}} \right) = -\frac{1}{2} g^{r\rho} \partial_0 g_{\rho m} \tag{43}$$

Since we have a diagonal metric, the following must be true:

$$\Gamma^k_{0k} = -\Gamma^k_{0m} = \Gamma^m_{0k} \tag{44}$$

Again, because we have a diagonal metric, the only non-zero components of R_{00} will be when $k = \rho = m = r$ in the Christoffel symbols (note: if I really didn't get the correct answer, this would be my incorrect assumption made). This results in:

$$\begin{aligned} R_{00} &= -\partial_0 \Gamma^k_{0k} - \Gamma^k_{0m} \Gamma^m_{0k} \\ \implies R_{00} &= -\frac{1}{2} \partial_0 \left(g^{k\rho} \partial_0 g_{\rho k} \right) - \frac{1}{4} g^{k\rho} g^{m\rho} \partial_0 \left(g_{\rho m} + g_{\rho k} \right) \\ \implies R_{00} &= -\frac{1}{2} \partial_0 \left(g^{kk} \partial_0 g_{kk} \right) - \frac{1}{4} g^{kk} g^{kk} \partial_0 \left(g_{kk} + g_{kk} \right) \end{aligned} \tag{45}$$

Now turning to the $-\frac{1}{2} g_{\mu\nu} R$ term of the Einstein field equations and using the same assumption that the indices must be the same since this is a diagonal metric:

$$R = g^{\mu\mu} R_{\mu\mu}. \tag{46}$$

Writing out the Ricci tensor again:

$$R_{\mu\mu} = \cancel{\partial_0 \Gamma^0_{\mu 0}} - \cancel{\partial_0 \Gamma^0_{\mu 0}} + \Gamma^0_{\mu 0} \Gamma^m_{0m} - \Gamma^0_{\mu m} \Gamma^m_{00} \tag{47}$$

Inspecting the Christoffel symbols and occasionally adopting the notation that $\partial_0 g_{\mu\nu} \equiv g_{\mu\nu,0}$:

$$\Gamma^0_{\mu 0} = \frac{1}{2} g^{\alpha 0} \left(g_{\alpha\mu,0} + \cancel{g_{\alpha 0,\mu}} - \cancel{g_{\mu 0,\alpha}} \right) = -\frac{1}{2} g_{\alpha\mu,0} \tag{48}$$

$$\Gamma^m_{0m} = \frac{1}{2} g^{\alpha m} \left(\cancel{g_{\alpha 0,m}} + g_{\alpha m,0} - \cancel{g_{0m,\alpha}} \right) = \frac{1}{2} g^{\alpha m} \partial_0 g_{\alpha m} \tag{49}$$

$$\Gamma^m_{00} = \frac{1}{2} g^{\alpha m} \left(\cancel{g_{\alpha 0,0}} + \cancel{g_{\alpha 0,0}} - \cancel{g_{00,\alpha}} \right) = 0 \tag{50}$$

These three expression simplify the $R_{\mu\mu}$ Ricci tensor into:

$$R_{\mu\mu} = \Gamma^0_{\mu 0} \Gamma^m_{0m} = \left(-\frac{1}{2} \partial_0 g_{\alpha\mu} \right) \left(\frac{1}{2} g^{\alpha m} \partial_0 g_{\alpha m} \right) \tag{51}$$

Again making the assumption that all of the indices must be the same in order to avoid the zero components of the metric:

$$\begin{aligned} R_{\mu\mu} &= -\frac{1}{4} g^{\alpha m} \partial_0 \left(g_{\alpha\mu} g_{\alpha m} \right) \\ \implies R_{\mu\mu} &= -\frac{1}{4} g^{\mu\mu} \partial_0 \left(g_{\mu\mu} g_{\mu\mu} \right) \end{aligned} \tag{52}$$

Combining all of this into the Einstein field equation:

$$\begin{aligned} \frac{1}{8\pi G} \left(R_{00} - \frac{1}{2} g_{00} R \right) &= \rho(t) \\ \implies \rho(t) &= \frac{1}{8\pi G} \left(R_{00} + \frac{1}{2} R \right) \\ \implies \rho(t) &= \frac{1}{8\pi G} \left(-\frac{1}{2} \partial_0 \left(g^{\mu\mu} \partial_0 g_{\mu\mu} \right) - \frac{1}{2} g^{\mu\mu} g^{\mu\mu} \partial_0 \left(g_{\mu\mu} \right) + \frac{1}{2} g_{\mu\mu} \left(g^{\mu\mu} \frac{1}{4} g^{\mu\mu} \partial_0 \left(g_{\mu\mu} g_{\mu\mu} \right) \right) \right) \\ \implies \rho(t) &= \frac{1}{8\pi G} \left(-\frac{1}{2} \partial_0 \left(g^{\mu\mu} \partial_0 g_{\mu\mu} \right) - \frac{1}{2} g^{\mu\mu} g^{\mu\mu} \partial_0 \left(g_{\mu\mu} \right) + \frac{1}{8} g^{\mu\mu} \partial_0 \left(g_{\mu\mu} g_{\mu\mu} \right) \right) \end{aligned} \tag{53}$$

where $g_{\mu\mu} = -1 + 3t^{2/3}$ and $g^{\mu\mu} = -1 + 3t^{3/2}$

Chapter 4: Hamiltonian Mechanics including Constraints and Fields

Problem (4.1): Show that the Poisson brackets satisfy the Jacobi identity, that is, if f, g, h are three functions of phase space, then,

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$$

Use it to show that if one has two constants of motion, one can generate a (potentially) new constant of motion by considering the Poisson bracket.

Solution: I will carry this out with a straight-forward caveman proof:

$$\begin{aligned}\{\{f,g\},h\} = \sum_{i=1}^{N} \frac{\partial h}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right) - \frac{\partial h}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right) \\ - \frac{\partial h}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right) + \frac{\partial h}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)\end{aligned} \tag{54}$$

$$\begin{aligned}\{\{g,h\},f\} = \sum_{i=1}^{N} \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{\partial g}{\partial q_i}\frac{\partial h}{\partial p_i}\right) - \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{\partial g}{\partial p_i}\frac{\partial h}{\partial q_i}\right) \\ - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial g}{\partial q_i}\frac{\partial h}{\partial p_i}\right) + \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial g}{\partial p_i}\frac{\partial h}{\partial q_i}\right)\end{aligned} \tag{55}$$

$$\begin{aligned}\{\{h,f\},g\} = \sum_{i=1}^{N} \frac{\partial g}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{\partial h}{\partial q_i}\frac{\partial f}{\partial p_i}\right) - \frac{\partial g}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{\partial h}{\partial q_i}\frac{\partial f}{\partial p_i}\right) \\ - \frac{\partial g}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial h}{\partial q_i}\frac{\partial f}{\partial p_i}\right) + \frac{\partial g}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial h}{\partial q_i}\frac{\partial f}{\partial p_i}\right)\end{aligned} \tag{56}$$

Expanding these terms out:

$$\begin{aligned}\{\{f,g\},h\} = \sum_{i=1}^{N} \frac{\partial h}{\partial p_i}\frac{\partial g}{\partial p_i}\frac{\partial^2 f}{\partial^2 q_i} + \frac{\partial h}{\partial p_i}\frac{\partial f}{\partial q_i}\frac{\partial^2 g}{\partial q_i \partial p_i} - \frac{\partial h}{\partial p_i}\frac{\partial g}{\partial q_i}\frac{\partial^2 f}{\partial q_i \partial p_i} - \frac{\partial h}{\partial_i}\frac{\partial f}{\partial p_i}\frac{\partial^2 g}{\partial^2 q_i} \\ - \frac{\partial h}{\partial q_i}\frac{\partial g}{\partial p_i}\frac{\partial^2 f}{\partial p_i q_i} - \frac{\partial h}{\partial q_i}\frac{\partial f}{\partial q_i}\frac{\partial^2 g}{\partial^2 p_i} + \frac{\partial h}{\partial q_i}\frac{\partial g}{\partial q_i}\frac{\partial^2 f}{\partial^2 p_i} + \frac{\partial h}{\partial q_i}\frac{\partial f}{\partial p_i}\frac{\partial^2 g}{\partial p_i \partial q_i}\end{aligned} \tag{57}$$

$$\begin{aligned}\{\{g,h\},f\} = \sum_{i=1}^{N} \frac{\partial f}{\partial p_i}\frac{\partial h}{\partial p_i}\frac{\partial^2 g}{\partial^2 q_i} + \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\frac{\partial^2 h}{\partial q_i \partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial h}{\partial q_i}\frac{\partial^2 g}{\partial q_i \partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial p_i}\frac{\partial^2 h}{\partial^2 q_i} \\ - \frac{\partial f}{\partial q_i}\frac{\partial h}{\partial p_i}\frac{\partial^2 g}{\partial p_i \partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial q_i}\frac{\partial^2 h}{\partial^2 p_i} + \frac{\partial f}{\partial q_i}\frac{\partial h}{\partial q_i}\frac{\partial^2 g}{\partial^2 p_i} + \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\frac{\partial^2 h}{\partial p_i \partial q_i}\end{aligned} \tag{58}$$

$$\begin{aligned}\{\{h,f\},g\} = \sum_{i=0}^{N} \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial p_i}\frac{\partial^2 h}{\partial^2 q_i} + \frac{\partial g}{\partial p_i}\frac{\partial h}{\partial q_i}\frac{\partial^2 f}{\partial q_i \partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial p_i}\frac{\partial^2 h}{\partial^2 q_i} - \frac{\partial g}{\partial p_i}\frac{\partial h}{\partial q_i}\frac{\partial^2 f}{\partial q_i \partial p_i} \\ - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}\frac{\partial^2 h}{\partial p_i \partial q_i} - \frac{\partial g}{\partial q_i}\frac{\partial h}{\partial q_i}\frac{\partial^2 f}{\partial^2 q_i} + \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}\frac{\partial^2 h}{\partial p_i \partial q_i} + \frac{\partial g}{\partial q_i}\frac{\partial h}{\partial q_i}\frac{\partial^2 f}{\partial^2 p_i}\end{aligned} \tag{59}$$

And, by inspection, all of these terms thankfully cancel out with each other when added together, resulting in the statement:

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0. \tag{60}$$

Problem 4.4: The Lagrangian for a relativistic particle is given by $L = -m\sqrt{u^\mu u_\mu}$ where $u^\mu = dx^\mu/d\tau$ is the four-velocity. Notice that the action is just given by the integral of the proper time. Work out the Lagrange equations of motion and show that the Hamiltonian vanishes. Why should have one expected the Hamiltonian to vanish?

Solution: Starting from the Euler-Lagrange equation

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{dt}\left(\frac{\partial L}{\partial\left(\frac{dx^\mu}{d\tau}\right)}\right) = 0, \tag{61}$$

we first find that

$$\frac{\partial L}{\partial x^\mu} = 0 \tag{62}$$

since L does not depend on x^μ. We then compute

$$-\frac{d}{dt}\frac{\partial}{\partial\left(\frac{dx^\mu}{d\tau}\right)}\left(-m\sqrt{u^\mu u_\mu}\right) \equiv \frac{d}{dt}\frac{\partial}{\partial\left(\frac{dx^\mu}{d\tau}\right)}\left(-m\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\right)$$
$$= \frac{d}{dt}\frac{m}{2}\left(\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)^{-1/2} 2\eta_{\mu\nu}\frac{dx^\nu}{d\tau} \equiv \frac{d}{dt}\left(\frac{mu_\mu}{\sqrt{u^\mu u_\mu}}\right) = \frac{d}{dt}(mu^\mu) = 0. \tag{63}$$

Which is just the relativistic form of Newton's second law. This tells us that $p_\mu = mu_\mu$. We can then compute the Hamiltonian for the system using the Legendre transformation:

$$H = p_\mu u^\mu - L = mu_\mu u^\mu - m\sqrt{u^\mu u_\mu} \equiv m - m = 0. \tag{64}$$

This should have been expected because the Hamiltonian of a free particle just corresponds to its rest mass energy in its own frame.

Problem 4.5: The action for a scalar field is given by

$$S = -\frac{1}{2}\int d^4x\left(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2\right).$$

Work out Hamilton's equations of motion for it.

Solution: When we define

$$\pi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}, \tag{65}$$

Hamilton's equations of motion for fields become:

$$\dot{q}_i = \frac{\partial\mathcal{H}}{\partial p_i} \implies \partial_\mu\phi = \frac{\partial\mathcal{H}}{\partial\pi^\mu} \tag{66}$$

$$\dot{p}_i = -\frac{\partial\mathcal{H}}{\partial q_i} \implies \partial_\mu\phi^\mu = -\frac{\partial\mathcal{H}}{\partial\phi}. \tag{67}$$

Since we are given an action, which is always written in the form

$$S = \int d^4x\mathcal{L}(\phi, \partial_\mu\phi), \tag{68}$$

we can extract $\mathcal{L}(\phi, \partial_\mu\phi)$, take the Legendre transformation of it to find $\mathcal{H}$, and then find the equations of motion above. First, since

$$\mathcal{L}(\phi, \partial_\mu\phi) = \partial_\mu\partial^\mu\phi + m^2\phi^2, \tag{69}$$

we use this to compute

$$\pi^\mu = \partial^\mu\phi \tag{70}$$

then we have

$$\mathcal{H} = \pi^\mu\partial_\mu\phi - \mathcal{L} = \partial^\mu\phi\partial_\mu\phi - \partial_\mu\partial^\mu\phi + m^2\phi^2 = m^2\phi^2. \tag{71}$$

With this, we find Hamilton's equations of motion:

$$\frac{\partial\mathcal{H}}{\partial\pi^\mu} = 0 \tag{72}$$

$$-\frac{\partial\mathcal{H}}{\partial\phi} = -2m^2\phi. \tag{73}$$