Derivation of the Wave Equation

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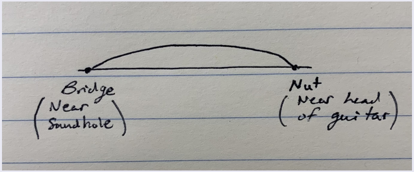
May 2023

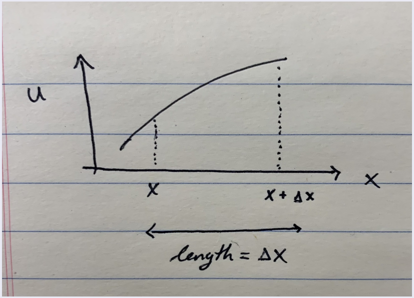
Setup

- Consider a string subject to the conditions:
  - Only vibrates in a fixed plane
  - Constant mass density ρ
  - Perfectly elastic and experiences no resistance
  - Transverse vibrations are small and occur in the x-axis (overall motion of the string depends only on the y-displacement and time)
- Such assumptions are not unrealistic. Consider a taut guitar string:



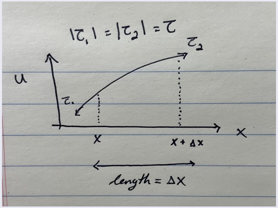
Closeup of String Segment

- Consider an infinitesimal region on the string of length Δx:

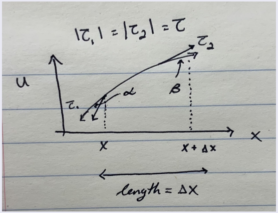


Relevant Forces

- String is taut $\implies$ tension force τ:

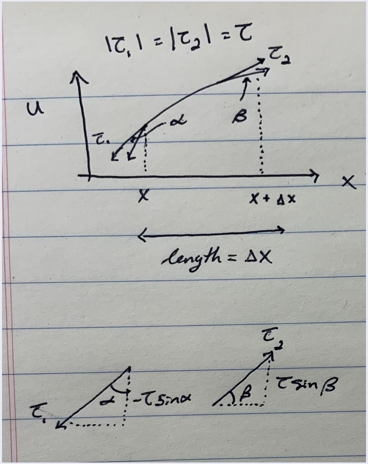


- Magnitude of τ must be equal on both ends but the positions we consider differ in angle $\implies$ two tension components τ_1 and τ_2 at angles α and β:



Relevant Forces Continued

- No motion in x-direction so we only consider the y-components of these tension forces:



Considering the Trigonometry of the String Segment

- Tension forces must balance because $|\tau_1| = |\tau_2| = \tau$ so:

$$\tau sin(\beta) = \tau sin(\alpha)$$

- By Newton's Second Law, the sum of tension forces must be equal to a mass multiplied by an acceleration:

$$-\tau sin(\alpha) + \tau sin(\beta) = F = ma$$

- Consider the small angle approximations $sin(\theta) \approx \theta, tan(\theta) \approx \theta$. $sin(\theta) = tan(\theta)$ under this approximation so make the replacement:

$$-\tau tan(\alpha) + \tau tan(\beta) = F = ma$$

Applying Newton's Second Law

- Recall in 1 dimension that:

$$F = ma = m\frac{d^2x}{dt^2}$$

- Because string depends on y-position and time $u(y, t)$:

$$F = ma = m\frac{\partial^2 u}{\partial t^2}$$

- Combining this with the previous trig replacement:

$$-\tau tan(\alpha) + \tau tan(\beta) = F = m\frac{\partial^2 u}{\partial t^2}$$

Calc 1 Definition of Tangents

- Recall from Calc 1 that a tangent line to a point is considered its derivative and the limit definition of a derivative is:

$$\lim_{x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{df}{dx}$$

- In this multivariable case $u(y,t)$, the variable in question here is the y-component so the tangent becomes a partial derivative
- Also remember that we consider a segment of string so evaluate these partials at both ends:

$$\tau tan(\alpha) = \tau \frac{\partial u}{\partial x}\bigg|_{(x)}$$

$$\tau tan(\beta) = \tau \frac{\partial u}{\partial x}\bigg|_{(x+\Delta x)}$$

Applying to Earlier Expression

- From original expression:

$$-\tau tan(\alpha) + \tau tan(\beta) = F = m\frac{\partial^2 u}{\partial t^2}$$

- Replace definitions for $tan(\alpha)$ and $\tan(\beta)$ from last slide:

$$-\tau \frac{\partial u}{\partial x}\bigg|_{(x+\Delta x)} + \tau \frac{\partial u}{\partial x}\bigg|_{(x+\Delta x)} = F = m\frac{\partial^2 u}{\partial t^2}$$

- Rearranging to get to a familiar form:

$$\tau(\frac{\partial u}{\partial x}\bigg|_{(x+\Delta x)} - \frac{\partial u}{\partial x}\bigg|_{(x+\Delta x)}) = m\frac{\partial^2 u}{\partial t^2}$$

Rearranging

- Notice that left side of the expression:

$$\tau\left(\frac{\partial u}{\partial x}\bigg|_{(x+\Delta x)} - \frac{\partial u}{\partial x}\bigg|_{(x)}\right) = m\frac{\partial^2 u}{\partial t^2}$$

is again in the form for the definition of a derivative:

$$\lim_{x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{df}{dx}$$

without the Δx term in the denominator. Divide both sides by this Δx term so left side is turned into a second derivative:

$$\tau\frac{\partial^2 u}{\partial x^2} = \frac{m}{\Delta x}\frac{\partial^2 u}{\partial t^2}$$

- Again, we have a multivariable expression so the derivative is a partial here as well.

Wave Equation Recovered

- Recall we are considering an infinitesimal of string and currently have:

$$\tau \frac{\partial^2 u}{\partial x^2} = \frac{m}{\Delta x} \frac{\partial^2 u}{\partial t^2}$$

- Where m refers to the overall mass of the string and Δx refers to the length of this small segment.
- Define $\frac{m}{\Delta x}$ to be the mass density ρ of the string.
- Divide τ to right side. $\frac{\tau}{\rho}$ has units of $\frac{[\text{length}]^2}{[\text{time}]^2}$ so call this c^2 (a velocity).

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

- Leaving you with the standard form of the wave equation [1] in one spatial dimension.
- Note that this could be easily extended to multiple dimensions since the system is linear.

Sources

[1] Nakhlé H Asmar. *Partial Differential Equations and Boundary Value Problems*. Upper Saddle River, N.J.: Prentice Hall, 2000.