Measuring a Pendulum's Deviation from Ideality

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(Dated: January 14 2024)

In this article, we consider the behavior of a pendulum when the small-angle approximation that is commonly used to idealize such a system is no longer valid by recording pendulum data with Pasco's ScienceWorkshop 750 device. We show that when one considers a pendulum at large oscillations, this idealized model cannot accurately describe the data collected and we show that an exact solution to the differential equation describing such a system, which requires a non-elementary function, can accurately model the data. This is done by fitting the idealized pendulum's model and this non-elementary function to the data collected and showing the idealized model's stark deviation from the data. We quantify the fit of both of these models to our collected data and find that $R^2 = .70$ for the non-elementary function and $R^2 = .29$ for the idealized model.

I. INTRODUCTION

The behavior of pendulums is ubiquitous in physics and is used to model various physical systems. Because the countless physical systems are often approximated as a pendulum, it is crucial to understand the behavior of pendulums under various conditions. To model the behavior of a pendulum, one must begin by considering the restoring torque of the pendulum $\tau = -mg\ell \sin\theta$ that causes the angular acceleration of the mass and the mass's moment of inertia $I = m\ell^2$ in Newton's second law $\tau = Id^2\theta/dt^2$. This differential equation becomes:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta \tag{1}$$

The ideal pendulum is a scenario considered when $\sin\theta \approx \theta$, which is the result of taking the first term in the Taylor series expansion for $\sin\theta$ and requires that the angles considered be "small". With this assumption, one can solve Eq. (1) and arrive at the result:

$$T_0 = 2\pi\sqrt{\frac{\ell}{g}} \tag{2}$$

We know from experience that not *all* pendulums oscillate with small angles, though, so we must consider the behavior of a pendulum when this assumption is not valid. When the small-angle approximation is not applied, there is no closed form solution to Eq. (1) in terms of elementary functions. The pendulum's period of oscillation, though, can be expressed by an integral of an elliptical function, defined as:

$$T(\theta_{\max}) = T_0 \frac{2}{\pi} K\left(\sin\left(\frac{\theta_{\max}}{2}\right)\right) \tag{3}$$

Where T_0 is the small-angle period and $K(x)$ is called the "complete elliptic integral of the first kind", which can be numerically evaluated.

In this paper, we use data collected from a pendulum to analyze the effectiveness of Eq. (3) in modelling the behavior of a pendulum when its oscillations are not small and quantify the system's deviation from the ideal, small-angle approximation. We show that Eq. (3) results in an R^2 value of .7 and Eq. (2) results in an R^2 value of .29 when fitted to the data, informing us of the insufficiency of the idealized pendulum when considered at large angles.

Though there are many, seemingly isolated, instances of the use of pendulums prior to the 17th century, it was Galileo's research on the properties of these systems, beginning in 1602 [1], that allowed us to recognize the utility of pendulums as clocks. In 1658, Christiaan Huygens designed and built the first pendulum clock, which was much more efficient and accurate for time keeping than the existing methods at the time [4]. Pendulums remained the most reliable method for timekeeping until the early 20th century, when quartz crystal oscillators were designed with a higher degree of accuracy than pendulums [2].

II. ANALYSIS

To analyze a pendulum's behavior for angles larger than those which Eq. (2) can be applied, we collect data from multiple pendulum trials, recording data for its maximum angle, period, and distance from the system's axis to its center of mass. As we collect data, we mark several trials where the distance from the pendulum axis to the center of mass is varied. So as to minimize the human error of these measurements, we collected the angle and period data with Pasco's ScienceWorkshop 750 device. The device is programmed to give some determined number of angle and time measurements per second, called the sampling frequency f, so we assumed that the error in our measurements would come from $1/f$, as there may not have been a data point available to us to record this data that was near the peak of the oscillations.

After collecting our data, we fit a function determined by Eq. (3) and a function determined by Eq. (2) to our data to analyze their explanatory accuracy. This resulted in the plot:

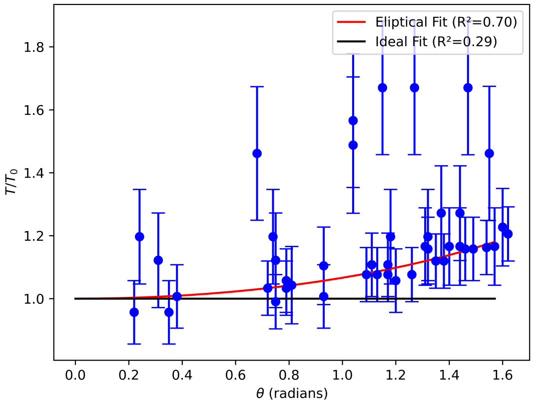

FIG. 1. Our recorded data plotted against the elliptical fit and the idealized fit. The R^2 values are shown in the legend of the plot. Our method for the computation of the error bars and R^2 values is provided in the Supplemental Information section.

Notice in this figure that the y-axis is scaled by Eq. (2), so one should expect that Eq. (2) remains a unit value in the plot.

This plot clearly shows that when one considers angles larger than those that can be captured by the small-angle approximation, Eq. (3) should be used to more accurately describe the behavior of the pendulum because Eq. (2) cannot provide an accurate model. This ability to accurately model the data collected is quantitatively measured by the R^2 values shown in the legend; where an R^2 value closer to one indicates that the model being considered represents a more accurate fit to your data. The difference between the R^2 value for the elliptical function and the idealized pendulum scenario emphasizes the conclusion that the latter cannot accurately model a pendulum when larger angles are considered.

III. CONCLUSIONS

In this work, we have determined that the idealized model for a pendulum, described by a small-angle approximation, is not valid when one considers a pendulum that swings at larger angles. We have found that in order to model the behavior of a pendulum in such conditions, an elementary function given by Eq. (3) must be used. We have quantified this deviation from the idealized scenario through the computation of an R^2 value, which indicates a given model's accuracy in describing the observed data. As predicted, we found that the R^2 value for the elliptical function indicates a much stronger correlation to the data than the idealized pendulum expression.

This result is important to consider when modeling physical systems as pendulums, as the idealized pendulum model may lead to a very inaccurate model and it may be common for these systems to exhibit larger oscillations. Because the small-angle approximation results from the first term in the Taylor series expansion for $\sin\theta$, a possibility for future research could include analyzing how including additional terms in this expansion into a pendulum model alters the resulting fit to this data. In this study, the period of a pendulum was only determined by considering a single oscillation. Extensions upon this work should also consider averaging multiple oscillations over time to more accurately determine the period of a pendulum, increasing the sampling rate of the monitor to reduce experimental uncertainties, and should consider examining the fit of the elliptical function to the data at the specific length (distance from pendulum axis to center of mass) values the data was recorded at to determine if the result of this study varies with length.

IV. SUPPLEMENTARY INFORMATION

The general formula for the uncertainty in a function with n variables, $f(x^0, x^1, \ldots, x^n)$, which all have uncertainties, $\delta x^0, \delta x^1, \ldots, \delta x^n$, is given as [3]:

$$\delta f = \sqrt{\sum_{i=0}^{n} \left(\frac{\partial f}{\partial x^i} \delta x^i \right)^2} \tag{4}$$

Since our function, T/T_0, where T_0 is given by Eq. (2), has two variables with errors, δl and δT, the error in our function is given by:

$$\delta\left(\frac{T}{T_0}\right) = \sqrt{\left[\delta T \frac{\partial}{\partial T}\left(\frac{T}{T_0}\right)\right]^2 + \left[\delta \ell \frac{\partial}{\partial \ell}\left(\frac{T}{T_0}\right)\right]^2} \tag{5}$$

Computing these derivatives results in:

$$\delta\left(\frac{T}{T_0}\right) = \sqrt{\left(\frac{\delta T}{T_0}\right)^2 + \left(\frac{T\sqrt{g}}{4\pi \ell^{3/2}}\right)^2} \tag{6}$$

The R^2 value is a metric used to determine how well a given model can predict a dataset. It is defined as [3]:

$$R^2 = 1 - \frac{\sum_{i=1}^{n}(y_i - f_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2} \tag{7}$$

Where y_i indicates a specific observation of some variable, $\bar{y_i}$ indicates the mean value of your observations, and f_i indicates the predicted value of these observations given the model being used.

REFERENCES

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