An Undergraduate Review of General Relativity and Cosmology

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General relativity is a subject that has too far-reaching of conclusions about the nature of reality and our universe to be only understood by specialists. If a non-specialist wishes to understand the theory without having to spend months or years studying it, they are often forced to resort to popular science articles that do not provide the theory's full depth and often simplify its details to the point of misrepresentation. In this report, I provide an introduction to general relativity at the undergraduate level that provides more insight into the theory than popular science treatments but does not utilize the level of rigor that is typically encountered in graduate-level treatments. I first give a justification for the hallmark of general relativity, the Einstein field equations, then show how they can be applied to describe a class of universes, and end by briefly reviewing the experimental results that confirm theoretical predictions made by these cosmological models. The intention of this report is not to properly educate a reader on general relativity and cosmology but to expose them to the basic theoretical structure of the matter and should serve as a starting point for the curious reader.

I. INTRODUCTION

First formulated in 1915 by Albert Einstein, the Einstein field equations are a set of highly coupled partial differential equations that describe the curvature of spacetime. Though an exact solution to these equations is difficult to construct due to the coupled components and the vast number of partial derivatives, physicists have been able to describe complex phenomena with them by making simplifying assumptions about the system they are attempting to model. To give the reader an understanding deeper than is given at the popular science level and in an attempt to display just how complex the field equations are, I will first give a justification for them from classical Newtonian gravity. I hope this exposition of the field equations is self-contained enough so that a reader can understand all of the terms that are hidden by the compact tensorial representation and so that no additional resources are needed to leave with a better understanding of how general relativity describes spacetime curvature.

To give an example of the sort of simplifying assumptions that are required to construct a solution to the field equations and to introduce the reader to a historical set of solutions to these equations that have been crucial to our understanding of the universe, I will then show how the Robertson-Walker metric can be derived. It isn't directly shown that the resulting metric provides a solution to the field equations in this work but resources where this is done are cited for the curious reader. I finish the article by briefly reviewing the experimental results that have come as a result of the Robertson-Walker metric to emphasize how far-reaching its consequences are and to provide evidence for the fact that the assumptions used to simplify this model are not unrealistic.

II. JUSTIFICATION FOR THE EINSTEIN FIELD EQUATIONS

A. Motivation from Newtonian Gravity

This section will provide motivation for the form of the Einstein field equations. This is in no way a full derivation of the Einstein field equations, as such an effort requires a great deal of classical mechanics and differential geometry that would be unnecessarily detailed at the level of this presentation[1]. This will largely follow the justification given in [4]. I will do my best to elucidate as many steps as I can and make this document as self-contained as possible. I will first begin by summarizing Newtonian gravitational theory and use its result as a sort of proto-gravitational field equation that will inform us of how to construct a more generalized, tensorial field equation.

To begin, when one considers the acceleration due to a gravitational force, the resulting expression for $\|\boldsymbol{g}\|$ is given by

$$\|\mathbf{g}\| = \frac{GM}{\|\mathbf{r}\|^2}. \tag{1}$$

If one then applies the divergence theorem to this vector field, $\boldsymbol{g}$, then one gets

$$\iiint_V (\nabla \cdot \mathbf{g}) \, \mathrm{d}V = \oiint_{S(V)} \mathbf{g} \cdot \hat{\mathbf{n}} \, \mathrm{d}S. \tag{2}$$

Since we are considering a gravitational field extending from a point source in three-dimensional space, $S(V)$ will be the surface area for a sphere. Since our gravitational acceleration vector, $\mathbf{g}$ will always act in the opposite direction of the normal vector to this sphere, $\hat{\mathbf{n}}$,

[1] A common derivation of the field equations starts from the consideration of the Einstein-Hilbert action, as shown in [1–3].

$\mathbf{g} \cdot \hat{\mathbf{n}} = -\|\mathbf{g}\|$. This considerations alter Eq. (2) to be of the form

$$\iiint_V (\nabla \cdot \mathbf{g})\, \mathrm{d}V = -4\pi G m. \tag{3}$$

Since the mass enclosed by this sphere can be written as a volume integral of the density, ρ, inside this sphere, this becomes

$$\iiint_V (\nabla \cdot \mathbf{g})\, \mathrm{d}V = -4\pi G \iiint_V \rho\, \mathrm{d}V, \tag{4}$$

which means by inspection that

$$\nabla \cdot \mathbf{g} = -4\pi G \rho. \tag{5}$$

We can now define the gravitational potential, $\boldsymbol{\phi}$, which is gravitational potential energy per unit mass, as

$$\|\boldsymbol{\phi}\| = \frac{Gm}{\|\mathbf{r}\|}. \tag{6}$$

By comparing Eq. (6) and Eq. (1), it's clear that $\boldsymbol{\phi}$ can be written as $\mathbf{g} = \nabla\boldsymbol{\phi}$ since r is the only spatial coordinate, which makes Eq. (3)

$$\nabla^2 \boldsymbol{\phi} = -4\pi G \rho. \tag{7}$$

This tells us that the gravitational potential field in Newtonian gravity is only caused by the mass density in that field.

B. Generalizing Newtonian Gravity

Given that we know the Newtonian gravitational potential to be a description able to describe the trajectory of objects in weak gravitational fields, it makes sense to use this theory of gravity as a base upon which to build a more generalized theory of gravity, using the tools of differential geometry.

Because the stress-energy tensor, $T^{\alpha\beta}$, offers a generalized description of mass density by encapsulating the energy density and momentum flux in addition to the mass density, it is reasonable to expect it to replace the ρ term in Eq. (7) when we generalize our theory of gravity. The stress-energy tensor is defined as

$$T^{\alpha\beta} = (\rho + p) U^\alpha U^\beta + p g^{\alpha\beta} \tag{8}$$

for a perfect fluid [4] where $\mathbf{U}$, the four-velocity of particles moving across some surface, is defined as

$$\mathbf{U} = \left(\frac{1}{\sqrt{1-v^2}}, \frac{v^x}{\sqrt{1-v^2}}, \dots \right) \tag{9}$$

in some frame of reference, $\mathcal{O}$, p is the momentum of the particles, ρ is now the mass-energy density, and $g^{\alpha\beta}$ is the inverse metric, which will be discussed later.

To further generalize Eq. (7), the Laplacian operator acting on the gravitational potential will be replaced by a general differential operator, $\mathbf{O}$, acting on the metric tensor $\mathbf{g}$, which is defined as

$$\mathbf{g} = g_{\alpha\beta} dx^\alpha dx^\beta \tag{10}$$

where the Einstein summation convention has been used to implicitly indicate a sum over the repeated α and β indices. The metric tensor is an object that captures properties of the intrinsic geometry of some manifold. We can then make the last step in generalizing Eq. (7) by replacing its $-4\pi G$ term with a general constant, k.

All of these generalizations of Eq. (7) result in

$$\mathbf{O}(\mathbf{g}) = k\mathbf{T}. \tag{11}$$

Since $\mathbf{T}$ is a rank-two contravariant tensor, indicated by the position of its indices in Eq. (8), we would expect that the differential operator $\mathbf{O}$ would produce a tensor of the same rank due to their equality. Since this operator is acting on the metric, we would additionally expect it to feature components $O^{\alpha\beta}$ that are combinations of $g_{\alpha\beta,\sigma\lambda}$, $g_{\alpha\beta,\sigma}$, and $g_{\alpha\beta}$ where the notation

$$g_{\alpha\beta,\sigma\lambda} \equiv \frac{\partial g_{\alpha\beta}}{\partial x^\sigma \partial x^\lambda} \tag{12}$$

has been used.

It is stated in [4] that any tensor of the form

$$O^{\alpha\beta} = R^{\alpha\beta} + \mu g^{\alpha\beta} R + \Lambda g^{\alpha\beta} \tag{13}$$

where μ and Λ are arbitrary constants will ensure that $\mathbf{O}$ contains components of $g_{\alpha\beta,\sigma\lambda}$, $g_{\alpha\beta,\sigma}$, and $g_{\alpha\beta}$. Though this choice seems arbitrary, it was likely chosen with the foresight of the results arrived at through an actual derivation of the Einstein field equations. Eq. (13) contains the Ricci tensor, $R^{\alpha\beta}$, which is defined as

$$R^{\alpha\beta} = \frac{1}{2} g^{\mu\sigma} \left(g^{\sigma\beta,\alpha\mu} - g^{\sigma\mu,\alpha\beta} + g^{\alpha\mu,\sigma\beta} - g^{\alpha\beta,\sigma\mu} \right) \tag{14}$$

and captures the curvature of a manifold by taking various second-order partial derivatives with respect to the metric where the raised indices in the metric terms indicate the inverted metric. For example, if one takes the metric to be a matrix with only non-zero diagonal components (which it often is), the inverse of this metric will just invert the diagonal components.

The R term in Eq. (13) is called the Ricci scalar, which more compactly captures the curvature of a manifold by only describing the average curvature at any point on that manifold. This is defined by summing the product of every term of the inverse metric by every term of the Ricci tensor

$$R = g^{\alpha\beta} R_{\alpha\beta}. \tag{15}$$

To continue, the strong equivalence principle, which states that all laws our physics are equivalent in all reference frames, that the gravitational acceleration will remain the same for any object experiencing it, and that the effects of gravity are indistinguishable from the effects of acceleration in flat spacetime, must be employed. The assumption of the strong equivalence principle and that it holds at all scales of the universe has not been experimentally proven to be incorrect. Since the principle must hold at all scales, analyzing a single point in spacetime will allow us to arrive at the Einstein field equations.

It is assumed that for a local inertial frame at a single point in spacetime, the metric will be approximately that of special relativity (flat spacetime), where the only differences are up to second order derivatives of the metric. Therefore, at some point $\mathcal{P}$, the metric will be

$$g_{\alpha\beta}(\mathcal{P}) = \eta_{\alpha\beta} \tag{16}$$

and its first-order derivative will be

$$\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}(\mathcal{P}) \equiv g_{\alpha\beta,\gamma}(\mathcal{P}) = 0 \tag{17}$$

because the spacetime is flat according to a local inertial frame where $\eta_{\alpha\beta}$ is the flat spacetime metric, defined as

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{18}$$

The covariant derivative, defined as

$$V^{\gamma}_{;\mu} = \frac{\partial V^{\alpha}}{\partial x^{\mu}} + V^{\beta}\Gamma^{\gamma}_{\beta\mu} \tag{19}$$

captures how a tensor changes from one point to another in any spacetime due to its dependence on $\Gamma^{\gamma}_{\beta\mu}$, the Christoffel symbols, which are defined by first-order derivatives of the metric

$$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\mu}} + \frac{\partial g_{\alpha\mu}}{\partial x^{\beta}} - \frac{\partial g_{\beta\mu}}{\partial x^{\alpha}}\right). \tag{20}$$

Since we are considering a flat spacetime metric at some point, $\mathcal{P}$, all of the terms in the Christoffel symbols vanish due to Eq. (17), leaving the covariant derivative $V^{\gamma}_{;\mu}$ directly equal to the regular partial derivatives, V^{α}_{μ}, in Eq. (19).

Since we are considering a single point, $\mathcal{P}$, in flat spacetime, it cannot be the case that the properties described by the stress-energy tensor, $T^{\alpha\beta}$, at that point change, meaning[2]

$$\frac{\partial T^{\alpha\beta}}{\partial x^{\mu}}(\mathcal{P}) = 0. \tag{21}$$

In fact, the statement must hold for any spacetime when you don't restrict the metric and must therefore consider the contribution of the Christoffel symbols, leading to the result that

$$T^{\alpha\beta}_{;\mu} = 0. \tag{22}$$

Returning to our developing generalization of the Newtonian gravitational field

$$R^{\alpha\beta} + \mu g^{\alpha\beta}R + \Lambda g^{\alpha\beta} = kT^{\alpha\beta} \tag{23}$$

since Eq. (22) is applied to the right-hand of this expression, the statement must also be applied to the left-hand side, meaning we must force

$$\left(R^{\alpha\beta} + \mu g^{\alpha\beta}R + \Lambda g^{\alpha\beta}\right)_{;\mu} = 0 \tag{24}$$

to be true.

It is a rule that for any basis, $g^{\alpha\beta}_{;\mu} = 0$ so we must ensure that

$$\left(R^{\alpha\beta} + \mu g^{\alpha\beta}R\right)_{;\mu} = 0. \tag{25}$$

With great convenience, this expression is exactly equal to the Einstein tensor, $G^{\alpha\beta}$ when $\mu = 1/2$, which is already defined such that $G^{\alpha\beta}_{;\mu} = 0$. This means that when one treats the left-hand side of Eq. (23) as $G^{\alpha\beta} + \Lambda g^{\alpha\beta}$, a valid tensor expression is derived, leaving us with

$$G^{\alpha\beta} + \Lambda g^{\alpha\beta} = kT^{\alpha\beta} \tag{26}$$

which is exactly the general form of the Einstein field equations[3]!

Because the Einstein field equations contain rank-two tensors in all terms, there are sixteen components that must be studied in order to derive a solution to the field equations[4]. Because these components are all highly coupled and contain *many* partial derivatives of the metric, one needs to employ many simplifying assumptions to their model or make use of modern supercomputers if they wish to describe a spacetime that is allowed by general relativity.

III. COSMOLOGICAL STRUCTURE AND EXPANSION

A. The Robertson-Walker Metric

After Einstein's theory of general relativity was published in 1916, two notable metric solutions to his field equations appeared in the following year. The first, published by Karl Schwarzschild in 1916, described the spacetime around a non-rotating spherical mass [5]. The second, published by Einstein himself in 1917, described a

[2] This statement is equivalent to the conservation of momentum and energy.

[3] In natural units ($G = c = 1$), it is shown that $k = 8\pi$ in [2, 3].

[4] There are actually ten *independent* components in the Einstein field equations but this does not reduce the immense complexity necessary to derive a solution to them.

finite and static universe with spherical spatial curvature [6].

A more sophisticated model of the universe was later independently developed by Howard Robertson and Arthur Walker in 1935, which has now become known as the Robertson-Walker metric [7, 8]. In their model, they describe a universe which is both homogeneous and isotropic. A homogeneous universe is one that, when averaged over large scales, appears to have an even distribution of matter and energy. An isotropic universe is one in which you would not be able to distinguish one direction in the sky from another. These assumptions about the *large-scale* structure of our universe have come to be known as the cosmological principle and, as will be discussed later, have been experimentally validated to great precision.

The cosmological principle therefore allows us to assume the spacetime of our universe to be of the form $\mathbf{R} \times \Sigma$, where $\mathbf{R}$ is the real line of the time dimension and Σ are the maximally-symmetric spatial dimensions [2, 3]. Because this space is assumed to be maximally symmetric, it is most reasonable to describe it as a sphere in spherical coordinates. To begin the derivation of the Robertson-Walker metric, we first derive the metric of flat spacetime in spherical coordinates.

For the transformation from Cartesian to spherical coordinates, the conversions $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, and $z = \cos\theta$ are necessary. The general metric in these coordinates is defined as the following collection of basis vector products:

$$g_{\alpha\beta} = \begin{pmatrix} \vec{e}_t \cdot \vec{e}_t & \vec{e}_t \cdot \vec{e}_r & \vec{e}_t \cdot \vec{e}_\theta & \vec{e}_t \cdot \vec{e}_\phi \\ \vec{e}_r \cdot \vec{e}_t & \vec{e}_r \cdot \vec{e}_r & \vec{e}_r \cdot \vec{e}_\theta & \vec{e}_r \cdot \vec{e}_\phi \\ \vec{e}_\theta \cdot \vec{e}_t & \vec{e}_\theta \cdot \vec{e}_r & \vec{e}_\theta \cdot \vec{e}_\theta & \vec{e}_\theta \cdot \vec{e}_\phi \\ \vec{e}_\phi \cdot \vec{e}_t & \vec{e}_\phi \cdot \vec{e}_r & \vec{e}_\phi \cdot \vec{e}_\theta & \vec{e}_\phi \cdot \vec{e}_\phi \end{pmatrix}. \quad (27)$$

Since this transformation from Cartesian to spherical coordinates does not depend on t, $\vec{e}_t$ will be of the form $\langle dt, 0, 0, 0\rangle$ and $\vec{e}_i$, where $i \in \{x, y, z\}$, will be of the form $\langle 0, ..., ..., ...\rangle$, which means that $\vec{e}_t \cdot \vec{e}_i = 0$. This turns our metric into

$$g_{\alpha\beta} = \begin{pmatrix} \vec{e}_t \cdot \vec{e}_t & 0 & 0 & 0 \\ 0 & \vec{e}_r \cdot \vec{e}_r & \vec{e}_r \cdot \vec{e}_\theta & \vec{e}_r \cdot \vec{e}_\phi \\ 0 & \vec{e}_\theta \cdot \vec{e}_r & \vec{e}_\theta \cdot \vec{e}_\theta & \vec{e}_\theta \cdot \vec{e}_\phi \\ 0 & \vec{e}_\phi \cdot \vec{e}_r & \vec{e}_\phi \cdot \vec{e}_\theta & \vec{e}_\phi \cdot \vec{e}_\phi \end{pmatrix}. \quad (28)$$

To compute $\vec{e}_r$, $\vec{e}_\theta$, and $\vec{e}_\phi$, the transformation

$$\vec{e}_{\alpha'} = \Lambda^{\beta}_{\alpha'} \vec{e}_\beta \quad (29)$$

will be used with $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, and $z = r\cos\theta$ where $\Lambda^{\beta}_{\alpha'}$ is the general coordinate transformation, defined as [4]

$$\Lambda^{\beta}_{\alpha'} \equiv \frac{\partial x^\beta}{\partial x^{\alpha'}}. \quad (30)$$

This transformation in spherical coordinates results in

$$\vec{e}_r = \frac{\partial x}{\partial r}\vec{e}_x + \frac{\partial y}{\partial r}\vec{e}_y + \frac{\partial z}{\partial r}\vec{e}_z \\ \implies \vec{e}_r = \sin\theta\cos\phi\vec{e}_x + \sin\theta\sin\phi\vec{e}_y + \cos\theta\vec{e}_z \quad (31)$$

$$\vec{e}_\theta = \frac{\partial x}{\partial \theta}\vec{e}_x + \frac{\partial y}{\partial \theta}\vec{e}_y + \frac{\partial z}{\partial \theta}\vec{e}_z \\ \implies \vec{e}_\theta = r\cos\theta\cos\phi\vec{e}_x + r\cos\theta\sin\phi\vec{e}_y - r\sin\theta\vec{e}_z \quad (32)$$

$$\vec{e}_\phi = \frac{\partial x}{\partial \phi}\vec{e}_x + \frac{\partial y}{\partial \phi}\vec{e}_y + \frac{\partial z}{\partial \phi}\vec{e}_z \\ \implies \vec{e}_\phi = -r\sin\theta\sin\phi\vec{e}_x + r\sin\theta\cos\phi\vec{e}_y. \quad (33)$$

By direct computation, it can then be shown that $\vec{e}_\theta \cdot \vec{e}_r = \vec{e}_r \cdot \vec{e}_\theta = \vec{e}_r \cdot \vec{e}_\phi = \vec{e}_\phi \cdot \vec{e}_r = \vec{e}_\theta \cdot \vec{e}_\phi = \vec{e}_\phi \cdot \vec{e}_\theta = 0$, which are notably all of the off-diagonal elements in $g_{\alpha\beta}$. It can also then be shown by direct computation that $\vec{e}_r \cdot \vec{e}_r = 1$, $\vec{e}_\theta \cdot \vec{e}_\theta = r^2$, and $\vec{e}_\phi \cdot \vec{e}_\phi = r^2 \sin^2\theta$. This results in the metric

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \quad (34)$$

It's important to note that while direct computation of $\vec{e}_t \cdot \vec{e}_t$ would result in 1, the negative sign is added to indicate the fact that we do not experience temporal symmetry[5].

Because the spatial components in Eq. (34) represent a sphere with a constant radius value, we can assume that this g_{rr} component can be replaced by some function $f(r)$, that provides a generalized description of the universe's spatial curvature and is only dependent on the radius. Again, due to symmetry, g_{rr} can only be a function of r because the object considered is spherically symmetric. If g_{rr} depended on the angles θ or ϕ, we would no longer have a maximally-symmetric object. Most textbooks on the subject assume $f(r) = e^{2\Lambda(r)}$ for mathematical convenience since this exponential term makes the derivation's subsequent partial derivatives more manageable.

These assumptions result in the modified metric

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & e^{2\Lambda} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \quad (35)$$

Schutz [4] goes on to compute the Einstein tensor and Ricci scalar, (15), for this developing metric so that it can be applied to the Einstein field equations.

[5] The existence of temporal symmetry would imply that the past is no different from the future. The signage of $g_{tt} = -1$ is an arbitrary choice, however. Some textbooks will often use the convention that $g_{tt} = 1$ and force the spatial components of the metric to carry negative signs. The variation of signage between the temporal and spatial components is necessary to indicate the directionality of time and its distinction from our spatial dimensions [9].

The Ricci scalar results in

$$R = \frac{2}{r^2}\frac{d}{dr}\left(1 - (re^{-2\Lambda})\right). \tag{36}$$

Because the Ricci scalar only represents the average curvature at a single point, it must be the case that R in the previous expression will just be some constant, k

$$k = \frac{2}{r^2}\frac{d}{dr}\left(1 - (re^{-2\Lambda})\right). \tag{37}$$

This expression can now be integrated in the following way:

$$\int \frac{kr^2}{2} dr = \int \frac{d}{dr}\left(1 - (re^{-2\Lambda})\right) dr \tag{38}$$

$$\implies \frac{kr^3}{6} + A = 1 - (re^{-2\Lambda}) \tag{39}$$

where A is just some constant of integration.

Since $g_{rr} = e^{2\Lambda}$, we can use the previous result to find a closed form solution for g_{rr}. Rearranging the previous expression results in

$$e^{2\Lambda} = g_{rr} = \frac{1}{1 - \frac{1}{6}kr^2 - \frac{A}{r}}. \tag{40}$$

Demanding the local flatness condition, given by Eq. (16), $g_{rr} = 1$ at some point, $\mathcal{P}$, where $r = 0$. In order for this to be satisfied, it must be true that $A = 0$. This results in

$$g_{rr} = \frac{1}{1 - \frac{1}{6}kr^2}. \tag{41}$$

Substituting this back into our metric results in

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{1}{6}kr^2} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{42}$$

We can generalize this result by introducing a positive scale factor, defined by convention to be $a^2(t)$, into the spatial components of the metric because this does not violate the symmetry we have required[6]

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2(t)}{1-\frac{1}{6}kr^2} & 0 & 0 \\ 0 & 0 & a^2(t)r^2 & 0 \\ 0 & 0 & 0 & a^2(t)r^2\sin^2\theta \end{pmatrix}. \tag{43}$$

[6] This $a(t)$ scale factor is interpreted to be the *expansion* or *contraction* of the universe. Our metric therefore describes a maximally symmetric object that can be expanding or contracting. The form of this scale factor will be explained in the following subsection.

Note that we cannot multiply the g_{tt} component by this scaling factor because we have demanded $g_{tt,t} = 0$; that our metric, describing the structure of the universe, not change over time.

Eq. (43) is the general Robertson-Walker metric. It is commonly expressed in the spacetime interval

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \tag{44}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$

B. Three Possible Geometries

It has been shown that if a universe is homogeneous and isotropic, it must have the metric described by Eq. (44). With the general metric of a homogeneous and isotropic universe defined, we should now turn our attention to its k-factor and consider what sort of geometries are described by it. This metric is shown to be homogeneous and isotropic for all k-values in [10, 11]. Notice that scale the r coordinate such that k is only allowed to take on three possible values: $-1, 0, 1$. For example, if $k = -3$, then r could be redefined as $\tilde{r} = \sqrt{3}r$ and $\tilde{a}(t) = a(t)/\sqrt{3}$ so the spacetime interval would remain unchanged

$$ds^2 = -dt^2 + \tilde{a}^2(t)\left[\frac{d\tilde{r}^2}{1-\tilde{r}^2} + \tilde{r}^2 d\Omega^2\right]. \tag{45}$$

This implies there are only three possible geometries we need to consider with the Robertson-Walker metric; $k = \pm 1$ and $k = 0$. As we will see, the consideration of these k-values is very important because they represent drastically different spatial curvature geometries.

Starting with the $k = 0$ case, the Robertson-Walker metric spacetime interval becomes

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2 d\Omega^2\right], \tag{46}$$

which is the metric for a flat spacetime represented in polar coordinates. Written just in terms of the metric, the $k = 0$ case is of the form

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t)r^2 & 0 \\ 0 & 0 & 0 & a^2(t)r^2\sin^2\theta \end{pmatrix}, \tag{47}$$

which is shown by Appendix A to be the metric of a four-dimensional sphere, called a three-sphere, with radius $a(t)$ and with *zero* curvature. The $k = 0$ case of the Robertson-Walker metric therefore describes a *flat* universe that is homogeneous and isotropic.

When considering the $k = 1$ case it's convenient to define a new variable

$$d\chi^2 = \frac{dr^2}{1-r^2}. \tag{48}$$

Integrating both sides of this expression results in

$$\chi = \sin^1(r), \tag{49}$$

which means that $r = \sin\chi$, transforming the Robertson-Walker metric into the form

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + \sin^2(\chi)d\Omega^2\right] \tag{50}$$

whose metric is therefore

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t)\sin^2\chi & 0 \\ 0 & 0 & 0 & a^2(t)\sin^2\chi\sin^2\theta \end{pmatrix}. \tag{51}$$

This is again shown by Appendix A to be the metric describing a three-sphere with radius $a(t)$ and with *positive* curvature. This model is called the *closed* or *spherical* Robertson-Walker metric, analogous to the surface of a balloon.

When considering the $k = -1$ case, it's similarly convenient to define the parameter:

$$d\chi^2 = \frac{dr^2}{1+r^2}. \tag{52}$$

Integrating both sides of this expression results in

$$\chi = \sinh^{-1}(r) \tag{53}$$

which means that $r = \sinh(\chi)$, transforming the Robertson-Walker spacetime interval into the form:

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + \sinh^2(\chi)d\Omega^2\right] \tag{54}$$

and resulting in the metric

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t)\sinh^2\chi & 0 \\ 0 & 0 & 0 & a^2(t)\sinh^2\chi\sin^2\theta \end{pmatrix}. \tag{55}$$

The spatial components of this metric again represent those of a three-sphere but the underlying geometry has changed. The $k = -1$ case instead describes a three-sphere with *negative* curvature and is called the *hyperbolic* or *open* Robertson-Walker metric, which analogizes our universe to the shape of a saddle.

It will be shown in the following section that we currently believe our universe to be most accurately described by the flat Robertson-Walker metric.

C. Expansion

Though the following derivation is typically carried out by solving for the scale factor, $a(t)$, in Eq. (43) by solving the Einstein field equations with a Robertson-Walker metric[7], a more intuitive and informative derivation for the purposes of this report will be carried out by considering a test mass on an expanding spherical surface[8].

Considering the Newtonian gravitational force experienced by a test mass at some distance away from the origin, r, of a three-dimensional sphere with mass density, ρ, results in

$$F = \frac{4}{3}\pi G\rho m r. \tag{56}$$

This means that the gravitational potential energy between the center of this sphere and the point is

$$U = -\frac{2}{3}\pi G\rho m r^2. \tag{57}$$

The kinetic energy of this test mass will just be in the traditional form

$$K = \frac{1}{2}m\dot{r}^2, \tag{58}$$

where $\dot{r} \equiv dr/dt$.

By the conservation of energy, we know that the sum of the kinetic and potential energy must not change over time; that it must equal some constant, C

$$C = \frac{1}{2}m\dot{r}^2 - \frac{2}{3}\pi G\rho m r^2. \tag{59}$$

Since we are assuming a homogeneous and isotropic universe that is uniformly expanding, the notion of comoving coordinates is employed to indicate the coordinate system that is carried along with this expansion and so is unchanged over time. Because the expansion is uniform, there will be a relationship between the "real coordinates" of an object, r (which we have defined as the distance of a test mass to the center of this expanding universe), and the comoving distances that are measured, x. This is written as

$$r = a(t)x, \tag{60}$$

where $a(t)$ is called the scale factor of the universe [13]. This relationship can be used to rewrite Eq. (59) in terms of this scale factor as

$$C = \frac{1}{2}m\dot{a}^2x^2 - \frac{2}{3}\pi G\rho m a^2x^2. \tag{61}$$

Solving for $\dot{a}/a$ by multiplying this expression by $2C/ma^2x^2$ results in

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{4\pi G\rho}{3} + \frac{2C}{ma^2x^2}. \tag{62}$$

[7] This can be seen in [2, 3, 12], for example.
[8] In this way, we are forcing $a^2(t)$ to be positive, which is an assumption that will be shown to be experimentally validated in the following section.

If we now consider this expression in natural units ($G = c = 1$) and define $k = -2C/mx^2$, this expression becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{4\pi\rho}{3} - \frac{k}{a^2}, \tag{63}$$

which is the first of the Friedmann equations. This describes the change in expansion of our universe over time with respect to its initial expansion rate[9]. It should be noted that the inclusion of the cosmological constant term, Λ, from the Einstein field equations changes very little of the results of this report's derivations. When Λ is considered in the full derivation of the first Friedmann equation, [13] tells us that we get the expression

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{4\pi\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3}. \tag{64}$$

The existence of this scale factor, $a(t)$ in the Robertson-Walker metric tells us that the Einstein field equations require that a homogeneous and isotropic universe either be uniformly expanding or contracting. The Einstein field equations do not allow homogeneous and isotropic universes to be static[10].

IV. EXPERIMENTAL VALIDATION

The derivations of the previous section assumed an expanding homogeneous and isotropic universe. After deriving the Robertson-Walker metric, Eq. (43), the claim that the $k = 0$ condition describes our universe was additionally made. In this section, a brief historical overview of how we have come to know such statements to be true will be given.

A. Evidence of Expansion

The theoretical discovery that the universe would need to be expanding to satisfy the Einstein field equations, Eq. (64), was first made by Alexander Friedmann in 1922 [14]. This was later independently discovered by the Belgian priest, George Lemaitre, in 1927 [15]. Lemaitre recognized that this dynamic picture of the universe should imply an explosive expansion from an extremely hot and dense initial state, a state he called the "primeval atom". In both papers, the change in the expansion of the universe over time with respect to its initial expansion was described by Eq. (64). This was defined to be some arbitrary constant, H_0

$$H_0 \equiv \frac{\dot{a}(t)}{a(t)}. \tag{65}$$

In 1929, the astronomer Edwin Hubble was measuring distances to nearby galaxies with stars that undergo very regular periods of fluctuation in their luminosity when he discovered that these galaxies were moving away from our own and the rate at which they were moving away was proportional to the distance they were away from us [16]. Hubble's first published measurements on this relationship were given in [16], which is reproduced here:

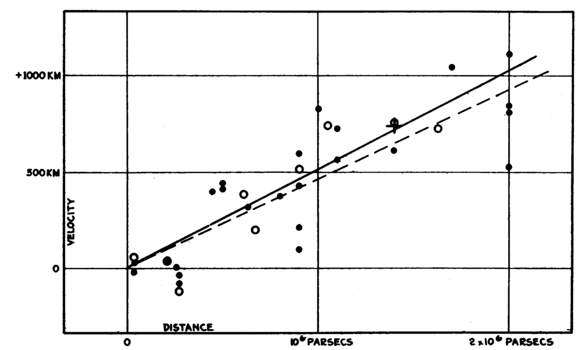

FIG. 1. The original plot of Hubble's redshift-distance relationship in his 1929 paper [16]. It shows that the rate at which galaxies are moving away from us is proportional to their distance away from us. This observation confirmed the theoretical predictions made by Alexander Friedmann in 1922 and George Lemaitre in 1927 that the universe is expanding and formed the basis of the Hubble law.

This observation was called the redshift-distance relation since the redshift of an object is related to its recessional velocity. The observation formed the basis for Hubble's law, written as

$$v = H_0 d, \tag{66}$$

where v is a galaxy's recessional velocity and d is its distance away from us. This experimentally confirmed the existence of the $\dot{a}(t)/a(t)$ term that appeared in Friedmann and Lemaitre's Robertson-Walker metric solutions to the Einstein field equations and informed us that it had a direct impact on the recessional velocities of objects in our universe.

B. Discovery of the Cosmic Microwave Background

In 1900, Max Planck was able to heuristically derive a formula to describe the observed spectrum of radiation from an idealized object with complete absorption of all energy incident upon it and with the perfect emission of light across the entire electromagnetic spectrum. This

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[9] It's a common misconception that the universe's expansion is constant. As we have shown, however, it is rather the *change* in this expansion over time with respect to its initial expansion that is constant.

[10] Again, we began this derivation with the assumption of a spherically expanding object. A more general derivation requires solving the Einstein field equations for a Robertson-Walker metric and solving for $a(t)$.

spectrum of the intensity of light $B(f,T)$ at a specific frequency f and temperature T is defined as [17]

$$B(f,T) = \frac{2hf^3}{c^2}\frac{1}{e^{hf/k_BT}-1}. \quad (67)$$

This description of "blackbody" radiation resolved the divergence at high frequencies predicted by classical physics and ushered in the quantum revolution by requiring that energy come in discrete packets. This result tells us that any object at a non-zero temperature will radiate particles of light, photons, in intensities given by the expression above.

During the 1930s and 1940s, cosmologists began considering how this law of blackbody radiation could be applied to the universe since we know the universe to contain electromagnetic radiation [18]. During the early universe, this radiation was expected to have been scattered and absorbed by the early matter, which would result in thermal radiation featuring a blackbody spectrum with a characteristic temperature. As the universe expands, the temperature of this radiation would decrease as the scale of the universe increases, meaning it would have become redshifted over time due to the expansion. It was first realized by Richard Tolman that as one goes further back in the history of the universe, this radiation would increasingly dominate the matter of the universe [19]. It was therefore theorized that this early radiation, later called the cosmic microwave background, could be observable with a powerful enough telescope.

This radiation was accidentally discovered by Arno Penzias and Robert Wilson while working with a large microwave horn antenna designed for satellite communications at Bell Labs in 1964. Although Penzias and Wilson originally thought the detected signal originated from pigeon droppings on the telescope's surface after discovering a nest in the telescope's antenna[11], they enlisted the help of Robert Dicke, who was actively searching for the cosmic microwave background radiation at Princeton University, when they learned about the theoretical predictions of this radiation [20]. When it was confirmed that they were detecting radiation from the early universe, Penzias and Wilson published their measurements of the cosmic microwave background in 1965 and Robert Dicke's group published their own in the same year [21, 22].

In 1989, the Cosmic Background Explorer (COBE) was launched with the intention of providing much more accurate measurements of the cosmic microwave background. The results of its initial measurements were released in a 1992 paper [23] which later won George Smoot and John Mather, who were the principal investigators for the satellite, the 2006 Nobel Prize. In this paper, the collaboration displayed the initial findings of COBE and showed the degree of accuracy to which the cosmic microwave background obeys the Planck spectrum, given by Eq. (67). The collaboration's initial data is shown in the following plot:

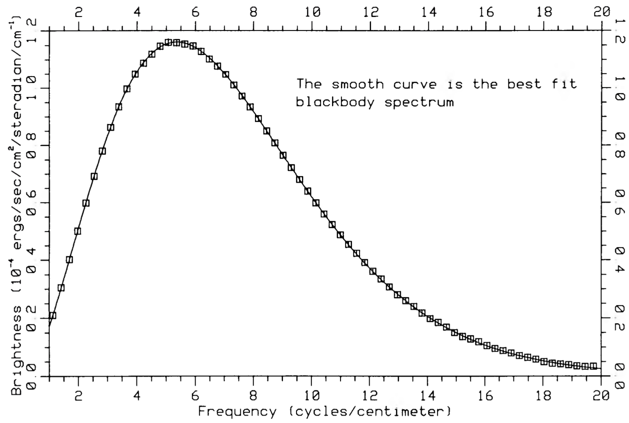

FIG. 2. The cosmic microwave background data collected by COBE, published in the collaboration's first paper [23]. The observed data almost exactly matches the theoretical curve at $2.728 \pm .004$ K, making this the "most perfect blackbody astronomically observed" [24].

C. Evidence of Homogeneity and Isotropy

In addition to offering the most precise measurement of the cosmic microwave background up to the 1990s, the COBE satellite also allowed cosmologists to accurately test the assumptions of homogeneity and isotropy that cosmological models such as the Robertson-Walker model, Eq. (44), depended on. Because the assumptions of homogeneity and isotropy are applied to the large-scale structure of the universe, it's clear that experimental verification of these assumptions requires advanced devices[12]. Prior to COBE, measurements of the cosmic microwave background's anisotropies were significantly limited by observational power due to the minute fluctuations in this radiation. COBE's precise measurements collected temperature fluctuations in the cosmic microwave background for two years and were able to discover temperature fluctuations from the mean on the order of 5×10^{-6} K, first published in 1992 [25].

To make even more precise measurements of the cosmic microwave background and its deviations from isotropy, the Wilkinson Microwave Anisotropy Probe (WMAP) was launched in 2001, and the Planck satellite in 2009. Among many other significant cosmological discoveries, these instruments have given theorists great confidence in their models' assumptions of homogeneity and isotropy.

[11] The pair later ensured that pigeons would not threaten their future observations by shooting all that were found near the telescope with a shotgun [20].

[12] Devices advanced enough to measure the large-scale structure of the universe existed prior to COBE but were very imprecise.

All measurements of the cosmic microwave background indicate that the universe, at very large scales, looks the same in all directions and has no preferred orientation. The increasingly detailed measurements of the cosmic microwave background are visualized in the following figure:

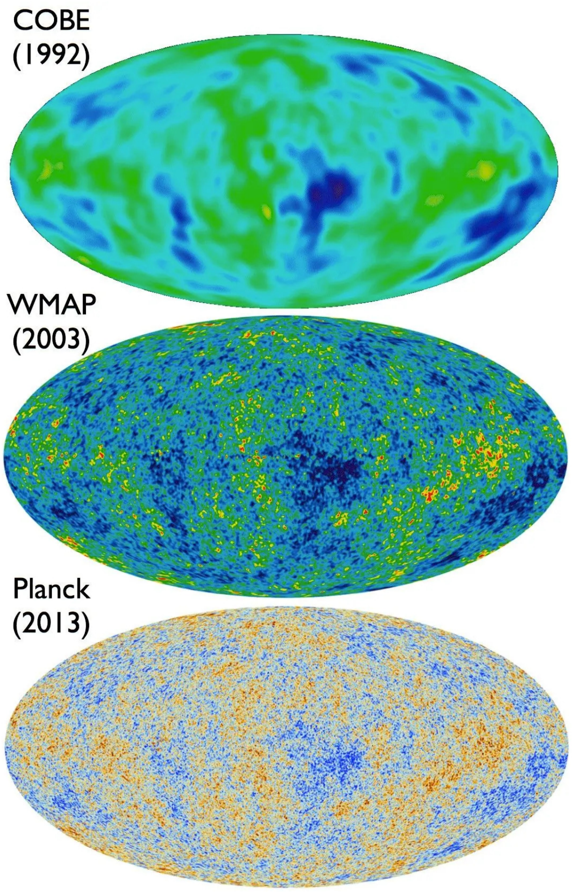

FIG. 3. Measurements of the cosmic microwave background made by COBE in 1990 [25], WMAP in 2003 [26], and the Planck satellite in 2018 [27] visualized. Images compiled by [28].

D. Evidence of a Flat Universe

In addition to allowing the assumptions of homogeneity and isotropy to be tested, the cosmic microwave background provides cosmologists a method to determine the geometry of our universe. Though there are many additional complications that need to be considered when determining the geometry of the universe, such as its energy density, it is enough for the purposes of this report to just provdie one of the ways in which the cosmic microwave background has allowed cosmologists to determine the structure of the universe.

Most important to determining the universe's structure from the cosmic microwave background is the regular fluctuations in the early universe's plasma. This plasma resulted from the fusing of baryonic matter (normal matter such as protons and electrons) and photons, brought about by the extremely high temperatures and density of the early universe. The periodic fluctuations of this plasma observed in the cosmic microwave background were caused by quantum fluctuations which led to variations in energy densities. These density perturbations caused pressure waves to propagate across the plasma as the early universe began to cool which became very periodic due to the characteristic speed of sound in the plasma. Because of this, these early density perturbations are given the name Baryon Acoustic Oscillations (BAOs) and are described in much greater detail in [29].

Studying these BAOs allows cosmologists to determine the structure of the universe because the distances these oscillations were allowed to travel set a length scale and a corresponding geometry to the early universe. This angular acoustic scale is captured by the ℓ_A parameter and the 2018 Planck satellite results [30] relate this to another representation, $100\theta_*$. The first release of the Planck satellite results reports this value to be 1.04097 ± 0.00046, which indicates that the early universe had a very flat geometry. Measurements of large-scale structure and galaxy distributions today additionally indicate a flat universe [31, 32]. When combined, these results indicate that our universe was flat in its early stages and its curvature has remained nearly flat over its expansion.

V. CONCLUSION

In this report, a brief introduction to general relativity and some of its consequences has been given. It's important to emphasize how complex the Einstein field equations are, meaning that exact solutions to them are difficult to construct. As has been shown, exact solutions can be arrived at by greatly approximating the system that is being attempted to be modeled (or by using the computational power of modern supercomputers). Both the far-reaching consequences of general relativity and the sort of simplifying assumptions necessary was emphasized in the example of the Robertson-Walker metric, which was one of the first exact solutions to the field equations. The Robertson-Walker metric depends on the assumption that the universe be homogeneous and isotropic at large scales and, as was shown later in the report, such assumptions accurately describe there universe we live in. This report additionally provided a primer on the theoretical motivation for why the field equations imply the existence of a dynamic universe, how we have come to know that the universe is expanding, and how we have come to know that the universe is flat.

This report intends that a reader familiar with the basics of undergraduate physics and mathematics can come away with a picture of how general relativity works at a level that stands above the popular science level of treatment and can come away with a better understanding of what the theory has to say about the nature of the universe, which is one of the most important consequences of the theory. A deeper understanding of the theory and its results will doubtlessly require the consultation of additional resources that expose the theory in its full glory. This report has provided some resources relevant to the material discussed where the detail is given at a more advanced level for the curious reader. This report should not serve as a proper instruction on general relativity but should expose the reader to the basic concepts of the theory and should serve as a starting point for the reader interested in learning general relativity.

Appendix A: Three-sphere Metric

If the following terms are defined:

$$w = r\cos\theta \tag{A1}$$
$$z = r\sin\chi\cos\theta \tag{A2}$$
$$x = r\sin\chi\sin\theta\cos\phi \tag{A3}$$
$$y = r\sin\chi\sin\theta\sin\phi, \tag{A4}$$

it can be shown that the coordinates (θ, ϕ, χ) define a three-sphere with radius r in the following way:

$$\begin{aligned} x^2 + y^2 + z^2 + w^2 = r^2\sin^2\chi\sin^2\theta\cos^2\phi + \\ r^2\sin^2\chi\sin^2\theta\sin^2\phi + \\ r^2\sin^2\chi\cos^2\theta + r^2\cos^2\chi. \end{aligned} \tag{A5}$$

Using the trigonometric identity $\cos^2\eta + \sin^2\eta = 1$ many times, it is found that

$$x^2 + y^2 + z^2 + w^2 = r^2, \tag{A6}$$

which shows that Eq. (A1) - (A4) define a three-sphere of radius r. Our choice of coordinates shows that this definition of a three-sphere is just a further generalization of spherical coordinates to a higher dimension.

To determine the metric of this three-sphere, we must compute the basis vectors of its coordinates (θ, ϕ, χ) using the regular transformation rule

$$\vec{e}_{\alpha'} = \Lambda^{\beta}_{\alpha'}\vec{e}_{\beta}. \tag{A7}$$

To compute the $\vec{e}_\theta$ basis vector, the following must therefore be performed:

$$\vec{e}_\theta = \frac{\partial x}{\partial\theta}\vec{e}_x + \frac{\partial y}{\partial\theta}\vec{e}_y + \frac{\partial z}{\partial\theta}\vec{e}_z + \frac{\partial w}{\partial\theta}\vec{e}_w, \tag{A8}$$

which results in

$$\begin{aligned} \vec{e}_\theta = r\sin\chi\cos\theta\cos\phi\vec{e}_x \\ + r\sin\chi\cos\theta\sin\phi\vec{e}_y - r\sin\chi\sin\theta\vec{e}_z. \end{aligned} \tag{A9}$$

Using the same method to determine $\vec{e}_\phi$ and $\vec{e}_\chi$ results in

$$\vec{e}_\phi = -r\sin\chi\sin\theta\vec{e}_x + r\sin\chi\sin\theta\cos\phi\vec{e}_y, \tag{A10}$$
$$\begin{aligned} \vec{e}_\chi = r\cos\chi\sin\theta\cos\phi\vec{e}_x + r\cos\chi\sin\theta\sin\phi\vec{e}_y \\ + r\cos\chi\cos\theta\vec{e}_z - r\sin\chi\vec{e}_w. \end{aligned} \tag{A11}$$

From the definition of the metric's components in terms of basis vectors,

$$g_{\alpha\beta} \equiv \vec{e}_\alpha \cdot \vec{e}_\beta, \tag{A12}$$

the metric of the three-sphere can therefore be computed to be of the form

$$g_{\alpha\beta} = \begin{pmatrix} r^2 & 0 & 0 \\ 0 & r^2\sin^2\chi & 0 \\ 0 & 0 & r^2\sin^2\chi\sin^2\theta \end{pmatrix} \tag{A13}$$

where $g_{\chi\chi} = r^2$. From this, it's clear that Eq. (50) contains a three-sphere with radius $a(t)$ in its spatial components.

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