Classical Mechanics Notes

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Contents

1 Euler-Lagrange Equation Derivation

We want a process for finding the minimal path between two points (x_1, y_1) and (x_2, y_2). To do this, we will imagine a scenario where a path $y(x)$ that minimizes the distance between two points is known, in addition to an "incorrect" path that doesn't do so, which we will call $Y(x) = y(x) + \alpha\eta(x)$ where $\eta(x)$ is a function that quantifies the difference between $y(x)$ and $Y(x)$ and α is some constant that must be added to ensure that $\int_{x_1}^{x^2} y(x)dx \geq \int_{x_1}^{x_2} Y(x)dx$. This is represented visually in Fig. 1.

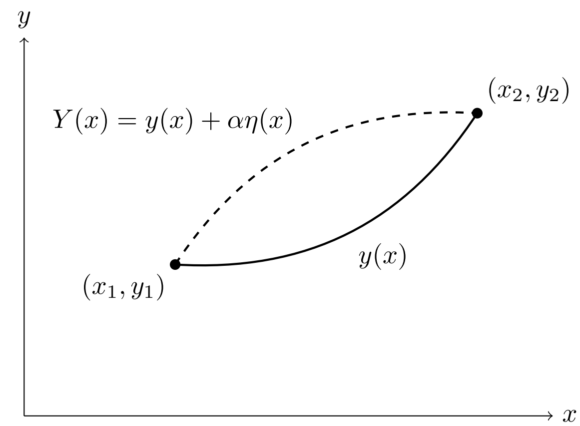

Figure 1: Demonstrating two arbitrary paths between point (x_1, y_1) and (x_2, y_2), where we know that $y(x)$ takes the optimal path between these points, and $Y(x)$ is any other path taken.

The quantity we want to minimize; the total path length of some $Y(x)$ between points (x_1, y_1) and (x_2, y_2), can be represented as an integral over some function that depends on $y(x), y'(x)$, and x:

$$S = \int_{x_1}^{x_2} f\big[y(x), y'(x), x\big]dx. \tag{1}$$

If we consider the non-optimal path in this expression, we will have an $S(\alpha)$ of the form

$$S(\alpha) = \int_{x_1}^{x_2} f\big[y(x) + \alpha\eta(x), y'(x) + \alpha\eta'(x), x\big] \tag{2}$$

since α is not integrated over the path; it is a free variable.

To find the optimal path between our points, we must therefore find where f achieves its minimum when we let $\alpha = 0$. To achieve this, we want to find where $\frac{\partial f}{\partial \alpha} = 0$. Because we have said that $f = f(y, y', x)$, we must use the chain rule to express $\partial_\alpha f$:

$$\frac{\partial f(Y, Y', x)}{\partial \alpha} = \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial \alpha} + \frac{\partial f}{\partial Y'}\frac{\partial Y'}{\partial \alpha} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha}, \tag{3}$$

and since $x \neq x(\alpha)$, $\partial_\alpha x = 0$. Using the expression for $Y(x)$, we then compute

$$\frac{\partial f}{\partial \alpha} = \eta(x)\frac{\partial f}{\partial Y} + \eta'(x)\frac{\partial f}{\partial Y'}. \tag{4}$$

since $\partial_\alpha\big(y(x) + \alpha\eta(x)\big) = \eta(x)$ and $\partial_\alpha\big(y'(x) + \alpha\eta'(x)\big) = \eta'(x)$.

To minimize Eq. (1) for $Y(x)$, we must now take

$$\frac{\partial S}{\partial \alpha} = \frac{\partial}{\partial \alpha}\int_{x_1}^{x_2} f\big[Y(x), Y'(x), x\big]\,dx = 0 \tag{5}$$

using our derived expression for $\partial_\alpha f$. Doing this, we find:

$$\frac{\partial S}{\partial \alpha} = \int_{x_1}^{x_2} \frac{\partial f(Y, Y', x)}{\partial \alpha}dx = \int_{x_1}^{x_2} \eta(x)\frac{\partial f}{\partial y} + \eta'(x)\frac{\partial f}{\partial y'}dx = 0. \tag{6}$$

We now rewrite the second term in this integral using integration by parts where $u = \partial_\alpha f$ and $dv = \eta' dx$. Applying this:

$$\int_{x_1}^{x_2} \eta'(x)\frac{\partial f}{\partial y'}dx = \eta(x)\frac{\partial f}{\partial y'}\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\frac{\partial f}{\partial y'}dx. \tag{7}$$

Because there is no difference between $y(x)$ and $Y(x)$ at the endpoints (x_1, y_1) and (x_2, y_2), this requires that $\eta(x_1) = \eta(x_2) = 0$, meaning

$$\int_{x_1}^{x_2} \eta'(x)\frac{\partial f}{\partial y'}dx = -\int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\frac{\partial f}{\partial y'}dx. \tag{8}$$

We now use this expression in Eq. (6) to find

$$\begin{aligned} \frac{\partial S}{\partial \alpha} = \int_{x_1}^{x_2} \frac{\partial f(Y, Y', x)}{\partial \alpha}dx &= \int_{x_1}^{x_2} \eta(x)\frac{\partial f}{\partial y} - \eta(x)\frac{d}{dx}\frac{\partial f}{\partial y'}dx = 0 \\ \implies \frac{\partial S}{\partial \alpha} &= \int_{x_1}^{x_2} \eta(x)\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)dx = 0. \end{aligned} \tag{9}$$

Since this must be fulfilled for any function $\eta(x)$, we can say

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0} \tag{10}$$

Eq. (10) is called the **Euler-Lagrange Equation**.

2 Lagrange's Equations for Unconstrained Motion

Goal: Write equations of motion in a framework that take the same form in any coordinate system and eliminates dependence on forces of constraint, such as the normal force, whose form are often not known exactly.

With the divine knowledge that such an approach will provide exactly what we're looking for, we define the following quantity:

$$\mathcal{L} := T - U, \tag{11}$$

called the **Lagrangian**. We let $T = T(\dot{x}, \dot{y}, \dot{z})$ since $T = \frac{1}{2}m\dot{\boldsymbol{r}}$ and assume $U = U(x, y, z)$ comes from a conservative force. We then discover that with such a definition,

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial U}{\partial x} \tag{12}$$

since only U depends on x, and then we use the definition of a conservative force to conclude that

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x. \tag{13}$$

It should be clear that this works for all Cartesian coordinates x, y, z.

Again using our definition of Eq. (11), we similarly discover

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \equiv p_x, \tag{14}$$

so if we then differentiate this expression with respect to time, we recover the definition of force:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\ddot{x} \equiv \dot{p}_x := F_x. \tag{15}$$

Since Eq. (13) and Eq. (15) both define F_x, setting them equal to each other recovers the Euler-Lagrange equation for x of the previous section:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x}, \tag{16}$$

which is called **Lagrange's equation**. It should again be clear that this argument works for all Cartesian coordinates x, y, z.

To remove the dependence of this method on Cartesian coordinates, we define the generalized coordinates q_i to specify a unique value of $\boldsymbol{r} = (x, y, z)$, $\boldsymbol{r} = (r, \theta, \phi)$, or any given coordinate system:

$$q_i = q_i(\boldsymbol{r}) \quad \text{for } i = 1, 2, 3. \tag{17}$$

In this way, we have

$$\boldsymbol{r} = \boldsymbol{r}(q_1, q_2, q_3), \tag{18}$$

and we make the same definitions for the time derivatives of these quantities:

$$\dot{q}_i = \dot{q}_i(\boldsymbol{v}) \quad \text{for } i = 1, 2, 3. \tag{19}$$

We can then write the Euler-Lagrange equations for our generalized coordinates $(q_i, \dot{q}_i)$ using the same argument we previously made for x, which results in the generalized form of Lagrange's equations:

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}}. \tag{20}$$

3 Proof of Euler-Lagrange Equation for System with Constraints

Goal: Prove that the Euler-Lagrange equation holds for a system experiencing constraint forces; such as a normal force or tension force. To make a more exact description of our goal, we wish to derive the Euler-Lagrange equation for a ***holonomic*** system; one where the number of degrees of freedom equals the number of generalized coordinates needed to describe the system.

To do this, we consider a system that is experiencing non-conservative constraint forces, $\boldsymbol{F}_{\text{cstr}}$, and regular non-constraint forces $\boldsymbol{F}$ that are conservative, such as the gravitational and spring force. Because our non-constraint forces are conservative, we can write them as being derived from a potential energy; $\boldsymbol{F} = -\nabla U(\boldsymbol{r}, t)$.

Similar to the first section, we consider any two points $(\boldsymbol{r}_1, t_1)$ and $(\boldsymbol{r}_2, t_2)$ and define $\boldsymbol{r}(t)$ to be the path actually realized in nature and $\boldsymbol{R}(t)$ to be any other paths that deviate from our physical path by some parameter $\boldsymbol{\epsilon}(t)$, which is now a vector since we are now considering a multi-dimensional system. As before, we write

$$\boldsymbol{R}(t) = \boldsymbol{r}(t) + \boldsymbol{\epsilon}(t) \tag{21}$$

where $\boldsymbol{\epsilon}(t_1) = \boldsymbol{\epsilon}(t_2) = 0$ since we have fixed the initial and final positions of our path (we *by definition* cannot have any deviations at our fixed points).

We then define the action integral for the general $\boldsymbol{R}(t)$ path as

$$S = \int_{t_1}^{t_2} \mathcal{L}(\boldsymbol{R}, \dot{\boldsymbol{R}}, t) dt \tag{22}$$

and define the action for the physical path as

$$S_0 = \int_{t_1}^{t_2} \mathcal{L}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) dt. \tag{23}$$

We can then define the difference δ between these two actions by

$$\delta S = S - S_0, \tag{24}$$

so if we show that $\delta S = 0$ to first order in $\boldsymbol{\epsilon}$, this tells us that for the general action S, the physical path taken will be where $\boldsymbol{R}(t) = \boldsymbol{r}(t)$. To derive the Euler-Lagrange equations for unconstrained motion, we must first show that $\delta S = 0$ to first order in $\boldsymbol{\epsilon}$.

First, the difference δ in the action is the integral difference in the Lagrangians on the two paths $\boldsymbol{r}(t)$ and $bR(t)$:

$$\delta \mathcal{L} = \mathcal{L}(\boldsymbol{R}, \dot{\boldsymbol{R}}, t) - \mathcal{L}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t). \tag{25}$$

Using our definition of $\boldsymbol{R}(t)$ from Eq. (21), we have

$$\delta \mathcal{L} = \mathcal{L}(\boldsymbol{r} + \boldsymbol{\epsilon}, \dot{\boldsymbol{r}} + \dot{\boldsymbol{\epsilon}}, t) - \mathcal{L}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t). \tag{26}$$

Because we defined the non-constraint forces on the system to be derivable from a potential energy, we can then rewrite each Lagrangian:

$$\mathcal{L}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) = T - U = \frac{1}{2}m\dot{\boldsymbol{r}}^2 - U(\boldsymbol{r}, t) \tag{27}$$

and

$$\mathcal{L}(\boldsymbol{r}+\boldsymbol{\epsilon}, \dot{\boldsymbol{r}}+\dot{\boldsymbol{\epsilon}}, t) = T - U = \frac{1}{2}m(\dot{\boldsymbol{r}}+\dot{\boldsymbol{\epsilon}})^2 - U(\boldsymbol{r}+\boldsymbol{\epsilon}, t). \tag{28}$$

Substituting these expressions into our difference $\delta\mathcal{L}$ gives us:

$$\delta\mathcal{L} = \frac{1}{2}m\left[(\dot{\boldsymbol{r}}+\dot{\boldsymbol{\epsilon}})^2 - \dot{\boldsymbol{r}}^2\right] - \left[U(\boldsymbol{r}+\boldsymbol{\epsilon}, t) - U(\boldsymbol{r}, t)\right]. \tag{29}$$

Expanding and simplifying our potential energy term leads to

$$\delta\mathcal{L} = \frac{1}{2}m\left[2\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{\epsilon}} + \dot{\boldsymbol{\epsilon}}^2\right] - \left[U(\boldsymbol{r}+\boldsymbol{\epsilon}, t) - U(\boldsymbol{r}, t)\right], \tag{30}$$

and we then drop the $\dot{\boldsymbol{\epsilon}}^2$ term since we are only considering the difference in actions to first order in $\boldsymbol{\epsilon}$.

We then use the generic approximation for a gradient of a scalar function

$$\nabla f \approx \frac{f(\boldsymbol{r}+\boldsymbol{\epsilon}) - f(\boldsymbol{r})}{\boldsymbol{\epsilon}} \tag{31}$$

to state that

$$\delta\mathcal{L} = m\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{\epsilon}} - \boldsymbol{\epsilon}\cdot\nabla U. \tag{32}$$

Substituting this into our expression for the difference in action δS, we have

$$\delta S = -\int_{t_1}^{t_2} m\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{\epsilon}} - \boldsymbol{\epsilon}\cdot\nabla U. \tag{33}$$

We then simplify the $m\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{\epsilon}}$ term using integration of parts:

$$\int_{t_1}^{t_2} m\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{\epsilon}}dt = m\dot{\boldsymbol{r}}\cdot\boldsymbol{\epsilon}\Big|_{t_1}^{t_2} - m\int_{t_1}^{t_2}\ddot{\boldsymbol{r}}\cdot\boldsymbol{\epsilon}dt = -m\int_{t_1}^{t_2}\ddot{\boldsymbol{r}}\cdot\boldsymbol{\epsilon}dt \tag{34}$$

where we have used the fact that $\boldsymbol{\epsilon}(t_1) = \boldsymbol{\epsilon}(t_2) = 0$.

Because $m\ddot{\boldsymbol{r}}$ represents the total force experienced by the object traveling along its physical path, and we have set our scenario up such that the object is subject to both non-conservative constraint forces $\boldsymbol{F}_{\text{cstr}}$ and conservative non-constraint forces $\boldsymbol{F}$; $\boldsymbol{F}_{\text{tot}} = \boldsymbol{F}_{\text{cstr}} + \boldsymbol{F} = \boldsymbol{F}_{\text{cstr}} - \nabla U$, we may now write our δS as

$$\begin{aligned}\delta S = -\int_{t_1}^{t_2} m\ddot{\boldsymbol{r}}\cdot\boldsymbol{\epsilon} - \boldsymbol{\epsilon}\cdot\nabla U &= -\int_{t_1}^{t_2}\boldsymbol{\epsilon}\cdot(\boldsymbol{F}_{\text{cstr}} - \nabla U) - \boldsymbol{\epsilon}\cdot\nabla U dt \\ &= -\int_{t_1}^{t_2}\boldsymbol{\epsilon}\cdot\boldsymbol{F}_{\text{cstr}}dt.\end{aligned} \tag{35}$$

Finally, because our constraint forces *always* act perpendicular to the direction of an object (consider the direction that normal and tensional force point in), we have that $\boldsymbol{\epsilon} \perp \boldsymbol{F}_{\mathrm{cstr}}$, so we have arrived at the desired result that $\delta S = 0$.

Because we have shown that the action integral is stationary at the correct path $\boldsymbol{r}(t)$ for a holonomic system, we have *not* proved that we can derive the Euler-Lagrange equations for a system expressed in *any* coordinate system; we have instead demonstrated that the Euler-Lagrange equations can be derived from an action for a system expressed in terms of its *generalized coordinates* q_i and $\dot{q}_i$, which take the constraints acting on a system into account.

4 Applications of the Euler-Lagrange Equations

Example 1: Find the equations of motion for a pendulum attached to the ceiling of an accelerating car using the Euler-Lagrange equations.

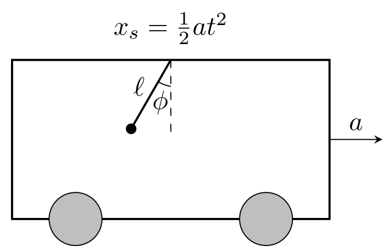

Figure 2: A figure of the scenario for Example 1.

To derive the equations of motion for this pendulum, we must first determine the Lagrangian for it. We first consider the kinetic energy term T.

Because $\dot{\boldsymbol{r}}^2 \perp\!\!\!\perp \dot{x}, \dot{y}$ by $\dot{\boldsymbol{r}}^2 = \dot{x}^2 + \dot{y}^2$, we must find the v_x and v_y components. For the v_x component, we have a contribution from the pendulum $\dot{x}_p$ and the acceleration $\dot{x}_c$ of the car. Because $x_p = \ell \sin\phi \implies \dot{x}_p = \ell\cos\phi\dot{\phi}$ and because $x_c = \frac{1}{2}at^2 \implies \dot{x}_c = at$. Combining these, we have $\dot{x} = \ell\cos\phi\dot{\phi} + at$. The y-component of the pendulum's velocity is only dependent on the pendulum's motion, so we have that $y = \ell\cos\phi \implies \dot{y} = -\ell\sin\phi\dot{\phi}$. We then have the following expression for $\dot{\boldsymbol{r}}^2$:

$$\dot{\boldsymbol{r}}^2 = \dot{x}^2 + \dot{y}^2 = (\ell\cos\phi\dot{\phi} + at)^2 + (\ell\sin\phi\dot{\phi})^2. \tag{36}$$

Expanding this in our expression for T:

$$T = \frac{1}{2}\left(\ell^2\dot{\phi}^2 + 2\ell at\cos\phi\dot{\phi} + (at)^2\right). \tag{37}$$

Since out potential energy is of the form $V = -mg\ell\cos\phi$, the full Lagrangian for our system is

$$\mathcal{L}(\phi, \dot{\phi}, t) = \frac{1}{2}\left(\ell^2\dot{\phi}^2 + 2\ell at\cos\phi\dot{\phi} + (at)^2\right) + mg\ell\cos\phi, \tag{38}$$

which we now will apply the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{39}$$

to so that was can derive the ball's equation of motion.

First computing the $\partial_\phi \mathcal{L}$ term:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \phi} &= \frac{\partial}{\partial \phi}\Big(m\ell at \cos\phi\dot{\phi} + mg\ell\cos\phi \Big) \\ &= -m\ell at\dot{\phi}\sin\phi - mg\ell\sin\phi. \end{aligned} \tag{40}$$

Then the $\partial_{\dot{\phi}} \mathcal{L}$ term:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} &= \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}m\ell^2\dot{\phi}^2 + mat\ell\cos\phi\dot{\phi} \right) \\ &= m\ell^2\dot{\phi} - mat\ell\cos\phi \end{aligned} \tag{41}$$

so we have that

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m\ell^2\ddot{\phi} - ma\ell\cos\phi - mat\ell\sin\phi\dot{\phi} \tag{42}$$

by the product rule.

Combining our expressions for $\partial_\phi \mathcal{L}$ and $\frac{d}{dt}\partial_{\dot{\phi}}\mathcal{L}$ for the Euler-Lagrange equation, we have that

$$\begin{aligned} m\ell^2\ddot{\phi} - ma\ell\cos\phi - mat\ell\sin\phi\dot{\phi} &= -m\ell at\dot{\phi}\sin\phi - mg\ell\sin\phi \\ \implies m\ell^2\ddot{\phi} - ma\ell\cos\phi &= -mg\ell\sin\phi \\ \implies \ddot{\phi} &= \frac{a}{\ell}\cos\phi - \frac{g}{\ell}\sin\phi. \end{aligned} \tag{43}$$

Notice that when $a = 0$, we recover the familiar equation of motion for simple harmonic motion.

5 Two Body System Equations of Motion

When the center of mass between two objects is defined;

$$\boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{M}, \tag{44}$$

one can write the Lagrangian of the system as

$$\mathcal{L} = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}\mu\boldsymbol{r}^2 - U(\boldsymbol{r}) \tag{45}$$

by additionally defining $\boldsymbol{r} := \boldsymbol{r}_1 - \boldsymbol{r}_2$ and the reduced mass as

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{46}$$

Notice that $\mathcal{L}$ has one term that depends on the center of mass position $\boldsymbol{R}$ and two terms that depend on the relative position of the two objects $\boldsymbol{r}$. This means we can separate our Lagrangian into two separate components

$$\mathcal{L} = \mathcal{L}_{\text{cm}} + \mathcal{L}_{\text{rel}} \tag{47}$$

If we wish to find the equations of motion for this system and physically understand what each of them tell us, we can use this allowed Lagrangian split, and solve the equations of motion for each component. To proceed, we first determine the equations of motion for the center of mass frame component to our full system's Lagrangian, $\mathcal{L}_{\text{cm}}$.

Because $\mathcal{L}_{\text{CM}}$ is independent of $\boldsymbol{R}$, we know that $M\dot{\boldsymbol{R}} = C$ from the Euler-Lagrange equations, which is a consequence of the conservation of angular momentum.

Moving to the $\mathcal{L}_{\text{rel}}$ component of our Lagrangian split, we find

$$\mu\ddot{r} = -\nabla U(\boldsymbol{r}) \tag{48}$$

after applying the Euler-Lagrange equations to our relative position variable $\boldsymbol{r}$ and its velocity $\dot{\boldsymbol{r}}$.

When analyzing the angular momentum of this system, Taylor goes on to show that

$$\boldsymbol{L} = \boldsymbol{r} \times \mu\dot{\boldsymbol{r}}. \tag{49}$$

Because angular momentum is conserved in this system, $\dot{\boldsymbol{L}} = 0$, which implies that $\boldsymbol{r} \times \dot{\boldsymbol{r}}$ is a constant, which then implies that the *direction* of $\boldsymbol{r} \times \dot{\boldsymbol{r}}$ is constant; so that the two-body system stays fixed on a plane in the center of mass frame.

Because our system in the center of mass frame stays fixed to a plane, it is natural to think of their relative position as being written in polar coordinates, where

$$\dot{\boldsymbol{r}} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}. \tag{50}$$

In polar coordinates, our Lagrangian in the center of mass frame (which only depends on $\boldsymbol{r}$) is rewritten as

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r). \tag{51}$$

When we apply the Euler-Lagrange equations to the ϕ variable of this Lagrangian, we find

$$\mu r^2\dot{\phi} = C, \tag{52}$$

and since $\vec{\ell} := \vec{r} \times m\vec{v}$, we have that $\ell = C$.

Now applying the Euler-Lagrange equation to the r variable, we find

$$\mu\ddot{r} - \mu r\dot{\phi}^2 = -\frac{\partial U}{\partial r} \tag{53}$$

where $\mu\ddot{r}$ is the familiar radial force experienced by the reduced mass and $\mu r\dot{\phi}^2$ is the angular component of the force, $\hat{F}_\phi$. To represent this in a more enlightening form, we rearrange Eq. (52) to state

$$\dot{\phi} = \frac{\ell}{\mu r^2}, \tag{54}$$

which means we can rewrite Eq. (53) as

$$\mu\ddot{r} = -\frac{\partial U}{\partial r} + \frac{\ell^2}{\mu r^3} = \frac{\partial U}{\partial r} + F_f, \tag{55}$$

which has the form of Newton's second law in one dimension for a system with mass μ and position r subject to a conservative force plus a "fictitious" force

$$F_f = \frac{\ell^2}{\mu r^3}, \tag{56}$$

which we can also express as being derivable from a fictitious potential energy $U_f(r)$:

$$U_f(r) = \frac{\ell^2}{2\mu r^2}. \tag{57}$$

Rewriting Eq. (55) in terms of this new fictitious potential energy, we have

$$\mu\ddot{r} = -\frac{d}{dr}\Big(U(r) + U_f(r)\Big), \tag{58}$$

and we finally define the effective potential as

$$U_{\text{eff}} = U(r) + U_f(r) = U(r) + \frac{\ell^2}{2\mu r^2}. \tag{59}$$

Finally multiplying both sides of Eq. (58) by $\dot{r}$, we get

$$\frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2\right) = -\frac{dU}{dt} \implies \frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2 - U(r)\right) = 0. \tag{60}$$

If we recognize $\frac{1}{2}\mu\dot{r}^2$ as the kinetic energy of the reduced system and $U(r)$ as its potential energy, we have shown that the system's total energy is conserved.