- **Objective:** Prove that spacetime intervals are invariant under Lorentz transformations.

Recall that a distance, or inteveral, between two events in spacetime is defined as:

$$ds^2 = -(cdt)^2 + dx^2 + dy^2 + dz^2 \tag{1}$$

and that the hyperbolic representation of a Lorentz boost in the x-direction (the Lorentz transformation only applied in the x-direction. This is done for mathematical simplicity but can be extended to any direction because the transformation is linear) can be written as:

$$\Lambda^{\mu'}_{\mu} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

where ϕ is a parameter, ranging from $-\infty$ to ∞, given by $\phi = \tanh^{-1}(v)$.

I will first represent it as the following column matrix and set $c = 1$ for simplicity in this derivation:

$$(dx^{\mu})^2 = \begin{pmatrix} -dt^2 \\ dx^2 \\ dy^2 \\ dz^2 \end{pmatrix} \tag{3}$$

where $(dx^0)^2 = -dt^2$ and so on.

To transform this interval, it can be multiplied by Eq. (2):

$$(dx^{\mu'})^2 = \Lambda^{\mu'}_{\mu}(dx^{\mu})^2 = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -dt^2 \\ dx^2 \\ dy^2 \\ dz^2 \end{pmatrix} \tag{4}$$

This results in the matrix:

$$(dx^{\mu'})^2 = \begin{pmatrix} \cosh\phi dt^2 - \sinh\phi dx^2 \\ -\sinh\phi dt^2 + \cosh\phi dx^2 \\ dy^2 \\ dz^2 \end{pmatrix} \tag{5}$$

Recalling that $\phi = \tan^{-1}(v)$, $(dx^{\mu'})^2$ can be simplified down to:

$$(dx^{\mu'})^2 = \begin{pmatrix} \gamma^2(dt^2 - v^2dx^2) \\ \gamma^2(dx^2 - v^2dt^2) \\ dy^2 \\ dz^2 \end{pmatrix} \tag{6}$$

where $\gamma = 1/\sqrt{1-v^2}$.

So to show that the spactime interval is invariant under the Lorentz transformation, we need to show that $-(dt')^2 + (dx')^2 = -dt^2 + dx^2$:

$$\begin{aligned} -(dt')^2 + (dx')^2 &= \gamma^2dt^2 - v^2\gamma^2dx^2 + \gamma^2dx^2 - v^2\gamma^2dt^2 \\ &= \gamma^2\left(1-v^2\right)dt^2 + \gamma^2\left(1-v^2\right)dx^2 \\ &= dt^2 + dx^2 \\ \implies -(dt')^2 + (dx')^2 &= dt^2 + dx^2 \end{aligned} \tag{7}$$

And since $(dy')^2 = dy^2$ and $(dz')^2 = dz^2$, we have shown that the spacetime interval is invariant under Lorentz transformations: $(ds')^2 = ds^2$