This derivation will be an exercise in extracting the equations of motion for a system using the Euler-Lagrange equations when one is handed them in the compact notation of a Lagrange density. I will first show how the Euler-Lagrange equations are arrived at in field theory.

The action, defined as the integral of a system's Lagrangian (the difference between its kinetic and potential energy) over time, is the fundamental quantity in classical mechanics since it encapsulates the dynamics of a system. In field theory, the Lagrangian L is written as a spatial integral of a Lagrange density $\mathcal{L}$, which is a function of a field ϕ and its derivatives $\partial_\mu \phi$. The Lagrange density is the more important quantity in field theory since it describes continuous systems and fields, whereas the Lagrangian only describes discrete systems with finite degrees of freedom. The action is therefore written as

$$S = \int L dt = \int \mathcal{L}(\phi, \partial_\mu \phi) d^4x. \tag{1}$$

If we let the field change infinitesimally from $\phi \to \phi' = \phi + \delta\phi$ over some time and assume that $\delta\phi = 0$ on the boundary of the region you're integrating over[1], we may apply the principle of least action to arrive at the system's dynamics. The principle of least action states that the path taken by a system will always be the one that minimizes the action. To minimize the action, we must take a *functional derivative* of Eq. (1) and set the result equal to zero. A functional derivative relates the infinitesimal change of a functional (a function that acts on functions) to a change in its functions. A functional J acting on a function f is defined as

$$J[f] = \int L(x, f(x), f'(x)) dx \tag{2}$$

so the functional derivative of J is defined as

$$\delta J = \int \left(\frac{\partial L}{\partial f} \delta f(x) + \frac{\partial L}{\partial f'} \frac{d}{dx} \delta f(x) \right) dx. \tag{3}$$

Applying this to the action, we have

$$\delta S = \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi) \right) d^4x. \tag{4}$$

Where we have applied Schwarz's theorem to commute ∂_μ with δ_μ; $\delta(\partial_\mu \phi) = \partial_\mu(\delta\phi)$, which we are allowed to do since ϕ is assumed to be continuous on Ω. We now turn our attention to the second term in this expression. Since

$$\int_\Omega \partial_\mu \left(\frac{\partial_\mu \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) d^4x = \int_\Omega \delta\phi \; \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) d^4x + \int_\Omega \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi) d^4x \tag{5}$$

we apply this to the previous expression and find

$$\int_\Omega \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi) d^4x = \int_\Omega \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) d^4x - \int_\Omega \delta\phi \; \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) d^4x \tag{6}$$

We then apply Stokes' theorem; $\int_\Omega d\omega = \int_{\partial\Omega} \omega$ to the second term of this expression and since we have specified that $\delta\phi = 0$ on $\partial\Omega$, we have:

$$\int_\Omega \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi) d^4x = \cancel{\int_{\partial\Omega} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \; d^3x} - \int_\Omega \delta\phi \; \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) d^4x \tag{7}$$

Returning to our action in Eq. (4) and setting it equal to zero:

$$\delta S = \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial \phi} - \int_\Omega \delta\phi \; \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right) \delta\phi \; d^4x = 0 \implies \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0. \tag{8}$$

With this, we have the Euler-Lagrange equations of motion for a field.

When learning how to work with Eq. (8), the classic Lagrange density one is asked to work out its equations of motion for is called the Klein-Gordon field. The Lagrange density for the Klein-Gordon field is defined as

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2. \tag{9}$$

[1] The assumption that $\delta\phi = 0$ on $\partial\Omega$ isn't necessary here but, as we will see, it makes the expression much easier to work with.

Why this is such a classic example is apparent when one compares the Klein-Gordon field to the Lagrangian for the classical harmonic oscillator

$$L = \frac{1}{2}mv^2 - \frac{1}{2}m\omega^2 x^2. \tag{10}$$

To work out the equations of motion for this field, we compute

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi \tag{11}$$

and

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} &= \frac{\partial}{\partial(\partial_\mu \phi)}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right) = \frac{\partial}{\partial(\partial_\mu \phi)}\left(\frac{1}{2}\partial_\mu\phi\eta^{\mu\nu}\partial_\nu\phi\right)\\ &= \frac{1}{2}\eta^{\mu\nu}\partial_\nu\phi\frac{\partial}{\partial(\partial_\mu \phi)}(\partial_\mu\phi) + \frac{1}{2}\partial_\mu\phi\eta^{\mu\nu}\frac{\partial}{\partial(\partial_\mu \phi)}(\partial_\nu\phi)\\ &= \frac{1}{2}\eta^{\mu\nu}\partial_\nu\phi + \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\delta^\mu_\nu = \frac{1}{2}\partial^\mu\phi + \frac{1}{2}\eta^{\mu\nu}\partial_\nu = \partial^\mu\phi.\end{aligned} \tag{12}$$

Plugging these results into the Euler-Lagrange equations, we get

$$\boxed{\left(m^2 + \partial_\mu\partial^\mu\right)\phi = 0} \tag{13}$$

which can also be written as

$$m^2\phi = \Box^2\phi \tag{14}$$

where $\Box^2$ is called the d'Alembertian operator, defined as

$$\Box^2 := \frac{\partial^2}{\partial t^2} - \nabla^2. \tag{15}$$

It can be easily verified that plane waves of the form

$$\phi = Ae^{i(k\cdot\vec{r}-\omega t)} \tag{16}$$

are valid solutions to Eq. (13).

To compute the Hamiltonian density $\mathcal{H}$,

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L}, \tag{17}$$

where $\pi := \pi = \partial\mathcal{L}/\partial\dot{\phi}$ for the Klein-Gordon field, we first compute

$$\pi := \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \equiv \frac{\partial \mathcal{L}}{\partial_0 \phi} = \frac{\partial}{\partial_0 \phi}\left(\frac{1}{2}(\partial_0\phi)^2 - (\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right) = \partial_0\phi \tag{18}$$

where the negative sign on the $(\nabla\phi)^2$ term came from the fact that we are working with the metric signature $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Then we have that

$$\mathcal{H} = \partial_0^2\phi - \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\partial_0^2\phi + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2. \tag{19}$$

Interpretation of $\mathcal{H}$ for Klein-Gordon field: The $\partial_0^2\phi$ tells us that the change of ϕ over time contributes to the energy density of the field, which is related to the kinetic energy of the field. The $(\nabla\phi)^2$ term tells us that the spatial variations of ϕ contributes to the field's energy density, which could be related to the potential energy of the field. The $m^2\phi^2$ term tells us that the mass of the field's quanta (its particles) contributes to the field's energy density. It's interesting to note the difference in signage between these terms; the terms with a positive sign are the terms that seem to be properties inherent to the field while the term with the opposite sign seems to be related to particle creation. It would make sense for the properties inherent to the field to positively contribute to the field's energy density while the property responsible for particle creation out of this field would negatively contribute to the energy density of the field, since it likely requires energy from the field to create particles.