- **Objective**: To derive Heisenberg's uncertainty principle by considering the Gaussian distribution of one variable and its Fourier Transform component.

To begin, consider an arbitrary Gaussian function of the form:

$$f(x) = ae^{\frac{-x^2}{2b^2}} \tag{1}$$

representing some distribution in x values where a and b are arbitrary constants. To ensure that this Gaussian represents a true probability distribution in the x variable, we take the squared magnitude of Equation 1:

$$|f(x)|^2 = a^2e^{\frac{-x^2}{b^2}} \tag{2}$$

Note that normalizing this Gaussian is not crucial to the derivation so the a term need not be considered. We know that a Gaussian distribution which is properly normalized and represents a probability distribution should appear in the form:

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{\frac{x^2}{2\sigma_x^2}} \tag{3}$$

so we can constrain the b value in Equation 2 to:

$$\frac{-x^2}{b^2} = \frac{-x^2}{2\sigma_x^2} \implies b = \sqrt{2}\sigma_x \tag{4}$$

Now that Equation 1 can be considered to be a proper probability distribution in the x variable with the constraint given by Equation 4, we can consider what the Fourier Transform of such a function will result in. Recall the Fourier Transform being of the form:

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx \tag{5}$$

Substituting Equation 1 into Equation 5:

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{\frac{-x^2}{2b^2}}e^{-ikx}dx = \int_{-\infty}^{\infty} e^{-(\frac{x^2}{2b^2}+ikx)} \tag{6}$$

Again note that we will not consider the a term from Equation 1 since this would only be required if desired to properly normalize our Gaussian function. A integral table involving definite integrals with quadratic functions in an exponential gives the generalized result:

$$\int_{-\infty}^{\infty} e^{-(\alpha x^2+\beta x)}dx = \sqrt{\frac{\pi}{\alpha}}e^{\frac{\beta^2}{4\alpha}} \tag{7}$$

So by inspection, $\alpha = \frac{1}{2b^2}$ and $\beta = ik$. This gives the result to Equation 6:

$$\hat{f}(k) = \sqrt{2b^2\pi}e^{\frac{-k^2b^2}{2}} \tag{8}$$

Just as we ignored the a term in Equation 1 and 2, we will again ignore the $\sqrt{2b^2\pi}$ term in Equation 6 since it is not necessary for our derivation. We can play the same game as was done at the beginning of this derivation by taking the squared magnitude of this Gaussian function in terms of a new variable, k, which also carries the b term from Equation 1. Taking the squared magnitude of Equation 8 (without the constant term) gives:

$$|\hat{f}(k)|^2 = e^{-k^2b^2} \tag{9}$$

We again know that if this should truly represent a probability distribution of the k variable, then it should follow the form:

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}e^{\frac{k^2}{2\sigma_k^2}} \tag{10}$$

meaning that we can set up an equality as was done in Equation 4 like so:

$$-k^2b^2 = \frac{-k^2}{2\sigma_k^2} \implies b = \frac{1}{\sqrt{2}\sigma_k} \tag{11}$$

Since we defined b in Equation 4, we can make a substitution, leading us to the result:

$$\sqrt{2}\sigma_x = \frac{1}{\sqrt{2}\sigma_k} \implies \sigma_x\sigma_k = \frac{1}{2} \tag{12}$$

- **Interpretation**: The result of Equation 12 tells us about the generalized limit of knowledge one can have about the spread in two variables that are related by the Fourier Transform. If we interpret σ_x to represent the probability distribution for a particle's position and σ_k to represent the probability distribution for the particle's corresponding wavenumber, Equation 12 tells us that we cannot have exact knowledge about the spread in both of these variables, meaning that we may have a high level of knowledge about the particle's spatial distribution but would therefore be limited in our knowledge about the particle's wavenumber distribution.

- **Heisenberg's uncertainty principle**: Recall the de Broglie relation, $p = \hbar k$, where p is a particle's momentum, $\hbar$ is Planck's constant and k is the particle's momentum. If we multiply $\hbar$ by the full expression in Equation 12, we get:

  $$\sigma_x \hbar \sigma_k = \frac{\hbar}{2} \tag{13}$$

  Which means we can apply the de Broglie relation to the left side of the expression and result in the most common formulation of Heisenberg's uncertainty principle:

  $$\sigma_x \sigma_p = \frac{\hbar}{2} \tag{14}$$

  Meaning that we are *fundamentally* limited in the amount of information we can have about both a particle's position and momentum.

With the understanding of particles as waves as well as particles that comes from quantum mechanics, is important to derive Heisenberg's uncertainty principle from the basis of the Fourier Transform to emphasize just how fundamental and natural this result is to this framework. The uncertainty principle is often treated as a mystical result from quantum mechanics, implying a limit on the precision of any measurement of a quantum mechanical system or that we cannot measure such a system without affecting it. While these statements are both true, they are not completely novel to the framework of quantum mechanics and are only the result of a much more fundamental truth about our interpretation of quantum mechanics.