ADM Formulation of Scalar-Tensor-Vector Gravity

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Abstract

General relativity has been an immensely successful theory of gravity, leading to the prediction and discovery of phenomena which has shaped our modern understanding of the universe. General relativity does not exist without challenges, however, due to the existence of singularities that are hypothesized to exist in the center of black holes and at the origin of our universe, and due to challenges that face the theory's cosmological paradigm. In attempting to search for theories of gravity beyond general relativity, writing proposed modified theories in the ADM formalism by applying a 3+1 decomposition to them would allow for a more intuitive understanding of the theories and may aid in their mathematical analysis. In this work, I apply the 3+1 decomposition to Scalar-Tensor-Vector Gravity (STVG), derive the theory's resulting constraint equations, and derive initial data conditions for the theory.

Contents

1 Introduction

General relativity, discovered by Albert Einstein in 1915 [3], has been an incredibly successful theory of gravity with far-reaching implications on the nature of reality that are still in the process of being understood. Given its continued ability to stand up to experimental verification[1], general relativity has become our modern paradigm for understanding the nature of gravity, and revived interest in formally approaching cosmological studies in the early twentieth century, a field which is continuing to experience success.

As is the case with any successful scientific paradigm, however, anomalies have begun to arise in theoretical and observational considerations of the theory. The most significant of which, the existence of singularities at the center of black holes and at the origin of the universe, seems to be so fundamental of a problem that various programs of quantum gravity have attempted to either unify all fundamental interactions of nature, as is done in the context of string theory, or quantize the gravitational field, as is the case in loop quantum gravity. The existence of singularities seems to be so fundamental to the theory of relativity that one cannot hope to solve this anomaly within the confines of the theory.

Another fundamental problem arising from astrophysical observations that conflict with the theoretical predictions made by general relativity is the proposed existence of dark matter. Dark matter was first discovered in 1933 when Fritz Zwicky noticed that the observed rate of rotation of stars around galaxies did not fit the rate of rotation predicted by general relativity. It was discovered that to account for this discrepancy, one needed to assume that galaxies contained a much more even radial distribution of mass, which is a result that *cannot* be predicted from the framework of general relativity.

Because the issue of dark matter arises from a conflict between astrophysical observations and the predictions made by general relativity, dark matter does not appear to be as fundamental of a problem to the theory of relativity as the existence of singularities is. As such, there has been development in the direction of *modifying* the underlying theory of relativity to account for the theoretical discrepancies with observation, which has been done with some success[2]. Although it is currently more favorable to ascribe a particle-nature to dark matter since proposed modified theories of gravity have not yet been able to simultaneously explain *all* observational anomalies, it is still reasonable to consider modified gravity as a possible explanation for at least *some* of these anomalies.

To support the development of theories of modified gravity, sophisticated analyses of a given theory's fundamental equations are necessary. One common method used to analyze any general four-dimensional theory is called ***3+1 decomposition***, which separates the full four-dimensional theory into spatial and temporal components. This method applied to general relativity has been especially useful in the development of relativistic simulations, such as the merging of

[1]For example, see [1, 9, 2, 8, 4]
[2]For example, see [6, 5]

black holes and the system's emission of gravitational waves[3]. One of the most successful theories of modified gravity called **scalar-tensor-vector (STVG) gravity** [7], commonly called MOdified Gravity (MOG), has not yet had this decomposition applied to its fundamental equations that describe the evolution of relativistic systems. To support the development of modified gravity as a possible alternative to dark matter, I present the 3+1 decomposition as applied to the equations of STVG.

To do this, I will first explain all of the relevant mathematics and general relativity in the following section. I will then move on to describe the general framework of 3+1 decomposition, and demonstrate how it is applied to the Einstein field equations. I do not intend for these sections to be an introduction to these topics, but they will provide essential definitions, basic explanations, and references when details beyond the scope of this thesis are required. The curious reader would use these sections as a guide to a more thorough understanding of the topics. I then finish by applying 3+1 decomposition to the field equations of SVTG and will derive the resulting constraint equations for the theory.

2 Differential Geometry and General Relativity

2.1 Notion of a Manifold

The framework of general relativity is written in the language of differential geometry, so providing a short outline of its relevant elements is necessary. We first define the topological space that general relativity is written on.

Definition 1. A **manifold** is a topological space $\mathcal{M}$ such that each point $p \in \mathcal{M}$ has an open neighborhood U which is homeomorphic to an open subset of a Euclidean space, $\mathbb{R}^n$.

It is natural to formulate general relativity on manifolds because this space allows us to define notions of distances, angles, and curvature. Manifolds also provide the flexibility to describe the region near each point as flat, even when the global topology is not necessarily flat, which is an assumption that, as we will see, plays a crucial role in general relativity. A visualization of a manifold is given in Fig. 1.

[3] Here is where I'll cite foundational papers in NR. Maybe Frans Pretorius?

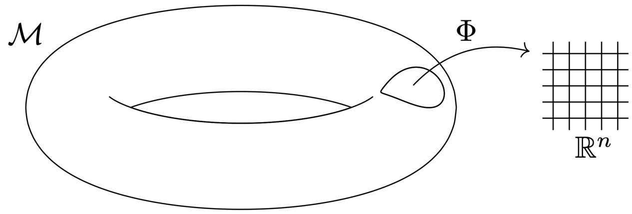

Figure 1: Space local to a point on a manifold will always appear Euclidean.

2.2 Vectors and Tensors on Manifolds

In special relativity, the globally flat geometry naturally defines a four-dimensional vector space structure. When moving to more general topologies, however, this vector space structure is lost. To retrieve these powerful tools in curved geometries, we recover vectors on manifolds in the limit of infinitesimal displacements about a point, which are called **tangent vectors**.

Definition 2. On a manifold $\mathcal{M}$, let $\mathscr{F}$ denote the collection of continuously differentiable functions from $\mathcal{M}$ into $\mathbb{R}^n$. The **tangent vector** V at a point $p \in \mathcal{M}$ is defined to be a map $V : \mathscr{F} \to \mathbb{R}^n$ which is (1) linear and (2) obeys the Leibniz rule:

1. $V(af + bg) = aV(f) + bV(g), \text{ for all } f, g \in \mathscr{F}, \text{ where } a, b \in \mathbb{R}^n;$
2. $V(fg) = f(p)V(g) + g(p)V(f).$

The collection of vectors tangent to a point $p \in \mathcal{M}$ is the tangent space $T_p(\mathcal{M})$. The tangent space at a point on a general manifold is illustrated in Fig 2.

In $\mathbb{R}^n$, vectors $V = (V^1, \ldots, V^n)$ define the directional derivatives $V^\mu\,(\partial/\partial x^\mu)$ and vice versa[4], so it is natural to express tangent spaces in terms of the set of directional derivatives (∂_μ).

[4]Note that we are using the Einstein summation convention throughout, where a repeated index denotes summation over all its possible values; $V^\mu \frac{\partial}{\partial x^\mu} \equiv \sum_\mu V^\mu \frac{\partial}{\partial x^\mu}$ where $\mu \in \{t, x, y, z\}$ in relativity.

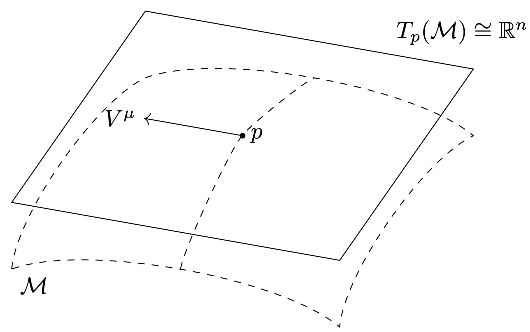

Figure 2: A vector V^μ drawn in the tangent space $T_p(\mathcal{M})$.

With tangent spaces allowing us to describe the set of all possible directions one can move locally on a manifold, we now must introduce the tools for more precisely describing how these directions are measured. We first define linear forms.

Definition 3. A **linear form** at each point $p \in \mathcal{M}$ is the linear mapping $\boldsymbol{\omega}$ such that

$$\boldsymbol{\omega} : T_p(\mathcal{M}) \longrightarrow \mathbb{R} \tag{1}$$

for all $p \in T_p(\mathcal{M})$.

The set of linear forms at p constitutes its own vector space, which is called the ***dual space*** of $T_p(\mathcal{M})$, denoted by $T_p^*(\mathcal{M})$. Vectors defined in the dual space are called ***contravariant***.

Given the natural basis (∂_μ) of $T_p(\mathcal{M})$ associated with a set of coordinates (x^μ) on a manifold, there exists a unique basis for $T_p^*(\mathcal{M})$, denoted by $(\boldsymbol{d}x^\mu)$ such that

$$\langle \boldsymbol{d}x^\mu, \partial_\nu \rangle = \delta^\mu_\nu, \tag{2}$$

where δ^μ_ν, called the Kroceker-delta, is defined as $\delta^\mu_\nu = 0$ when $\nu \neq \mu$ and $\delta^\mu_\nu = 1$ when $\nu = \mu$. This basis $(\boldsymbol{d}x^\mu)$ is called the dual basis of (∂_μ). Note that the natural bases are not the only possible choice in the vector space $T_p(\mathcal{M})$. One could use a general basis $(\boldsymbol{e}_\mu)$ that is not related to any coordinate system on $\mathcal{M}$. Given $(\boldsymbol{e}_\mu)$, there then exists unique dual bases $(\boldsymbol{e}^\mu) \in T^*(\mathcal{M})$ such that

$$\langle \boldsymbol{e}^\mu, \boldsymbol{e}_\nu \rangle = \delta^\mu_\nu. \tag{3}$$

A smooth assignment of linear forms to each tangent space at points on a manifold is called a ***1-form***. Given a smooth scalar field f on $\mathcal{M}$, the canonical 1-form associated with it is the gradient of f, defined as

$$\nabla f = \frac{\partial f}{\partial x^\mu} \boldsymbol{d}x^\mu. \tag{4}$$

With our definitions of tangent spaces and linear forms, we are now ready to define the objects that will be fundamental to general relativity, tensors, which are a generalization of vectors and linear forms.

Definition 4. At a point $p \in \mathcal{M}$, a **tensor** of type (k, ℓ) with $(k, \ell) \in \mathbb{N}^2$ is a mapping

$$\boldsymbol{T} : \underbrace{T_p^*(\mathcal{M}) \times \cdots \times T_p^*(\mathcal{M})}_{k \text{ times}} \times \underbrace{T_p(\mathcal{M}) \times \cdots \times T_p(\mathcal{M})}_{\ell \text{ times}} \longrightarrow \mathbb{R} \tag{5}$$

that is linear with respect to each of its arguments. The natural number $k + \ell$ is called the rank of the tensor.

Given a basis $(\boldsymbol{e}_\mu)$ of $T_p(\mathcal{M})$ and the corresponding dual basis $(\boldsymbol{e}^\mu) \in T_p^*(\mathcal{M})$, we can expand any tensor $\boldsymbol{T}$ of type (k, ℓ) as

$$\boldsymbol{T} = \boldsymbol{T}^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_\ell} \boldsymbol{e}_{\mu_1} \otimes \cdots \otimes \boldsymbol{e}_{\mu_k} \otimes \boldsymbol{e}^{\nu_1} \otimes \cdots \otimes \boldsymbol{e}^{\nu_\ell}. \tag{6}$$

where the $\otimes$ operation is the tensor product, with components $X^\mu V^\nu$ for vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$.

With tangent spaces and general tensors defined, we can now move on to the key structure of a manifold, the ***metric*** $\boldsymbol{g}$, which is a (0,2) tensor that allows one to define distances and angles on a given manifold. A metric acts on vectors in the tangent space $T_p(\mathcal{M})$ by

$$\boldsymbol{g} : T_p(\mathcal{M}) \times T_p(\mathcal{M}) \longrightarrow \mathbb{R}, \tag{7}$$

is symmetric; $\boldsymbol{g}(u, v) = \boldsymbol{v}, \boldsymbol{u}$, and is non-degenerate; there exists a non-zero $\boldsymbol{v} \in T_p(\mathcal{M})$ such that $\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = 0$ for all $\boldsymbol{u} \in T_p(\mathcal{M})$.

In a given basis $(\boldsymbol{e}_\mu)$ of $T_p(\mathcal{M})$, the components of $\boldsymbol{g}$ is the matrix $\boldsymbol{g}_{\mu\nu}$ defined by the expression for a general tensor given in Eq. (5) with $(k, \ell) = (0, 2)$:

$$\boldsymbol{g} = g_{\mu\nu} \boldsymbol{e}^\mu \otimes \boldsymbol{e}^\nu. \tag{8}$$

For any vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, we then have that

$$\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = g_{\mu\nu} u^\mu v^\nu. \tag{9}$$

2.3 Connections on Manifolds and Curvature

(Levi-Civita connection, covariant derivatives, and curvature)

In relativity and related theories, the space of physical events is represented by a Lorentzian manifold $\mathcal{M}$. Unlike a Riemannian manifold, which requires the metric to be positive definite, a Lorentzian manifold allows the metric to have an indefinite signature, meaning inner products can be positive, negative, or zero. This feature preserves the causal structure of spacetime by distinguishing between timelike, spacelike, and lightlike intervals[5]. Additionally, a Lorentzian

[5]See Appendix B for an explanation of causality in relativity.

manifold has a tangent space at each point that is locally homeomorphic to flat spacetime, described by the Minkowski metric $\eta_{\mu\nu}$. The tangent space represents the set of possible directions for movement tangential to a point, allowing the definition of differential objects like gradients on the manifold. The local flatness of the Lorentzian manifold at each point ensures that, on small scales, spacetime appears flat, preserving the validity of special relativity locally.

The manifold is endowed with a metric $g_{\mu\nu}$, which smoothly assigns a value to each point p on $\mathcal{M}$. This smoothness is crucial for ensuring the continuity and differentiability of the manifold. The metric allows the calculation of lengths and angles on $T_p\mathcal{M}$, the tangent space at each point. The metric $g_{\mu\nu}$ encodes the local geometry of the manifold near a point.

The tangent space at a point provides a way to approximate the manifold by flat spacetime at that point. Specifically, at any point p on the manifold, the tangent space $T_p\mathcal{M}$ locally resembles flat spacetime:

$$(\mathcal{M}, g_{\mu\nu}) \xrightarrow{T_p(\mathcal{M})} (\mathbb{R}^{1,n}, \eta_{\mu\nu}). \tag{10}$$

Every Lorentzian manifold comes with a **Levi-Civita connection**, which is a specific type of affine connection that is associated with a Lorentzian manifold. An **affine connection** is a geometrical object that connects tangent spaces at points on a manifold so that derivatives can be taken on the manifold.

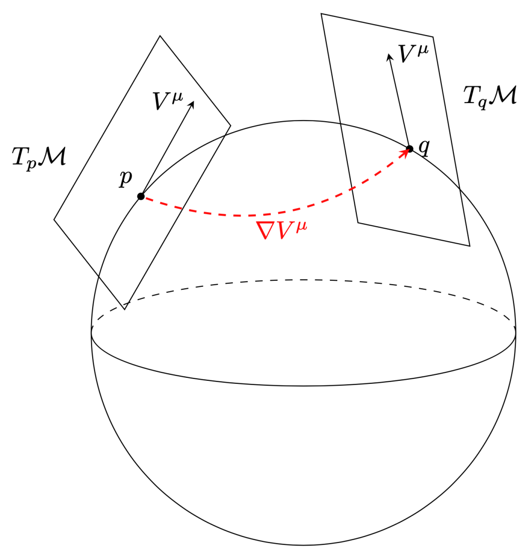

Figure 3: The affine connection from point p to q on a spherical manifold. The tangent spaces and a vector in them for each point are also shown.

This affine connection (just referred to as *connection*) is used to define the covariant derivative of a vector V^μ, defined as

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\lambda} V^\lambda \tag{11}$$

where the $\partial_\nu V^\mu$ term captures how the vector itself changes from p to q (as usual) and the $\Gamma^\mu_{\nu\lambda} V^\lambda$ term, called **Christoffel symbols**, captures how the shape of the manifold (V^μ's basis vectors) changes along the path. The Christoffel symbols are the coefficients of the connection. Since the Christoffel symbols capture how the manifold changes from one point to another, it shouldn't be surprising that their definition contains first derivatives of the metric;

$$.\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\sigma\mu} \left(\frac{\partial g_{\sigma\nu}}{\partial x^\lambda} + \frac{\partial g_{\sigma\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\sigma} \right). \tag{12}$$

In a flat spacetime, the metric would never change from one point to another, meaning $\Gamma^\mu_{\nu\lambda} V^\lambda = 0$, which means our covariant derivative would become

$$\nabla_\nu V^\mu = \partial_\nu V^\mu, \tag{13}$$

which is the regular definition of the gradient operator (so the covariant derivative is just the generalization of the gradient operator to curved spacetimes). If the covariant derivative of a vector along a path is zero, this means that the vector has not changed its orientation over the path. In this case, we say it has been parallel transported across the manifold.

Note that the shortest path on a manifold, called its **geodesic**, is described by the geodesic equation:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0 \tag{14}$$

where the x^μ terms are the coordinates on the manifold and τ is the proper time; the time measured by the observer moving along the geodesic.

To measure the **curvature of the manifold**, we must know how the metric, which, again, describes how the geometry near a point changes, itself changes over the manifold. Specifically, we must know how the geometry near each point changes and how these changes vary from point to point. The curvature of the manifold, which encapsulates how the manifold deviates from flatness, is described by the **Riemann curvature tensor**, defined as

$$R^\lambda_{\sigma\mu\nu} = \partial_\mu \Gamma^\lambda_{\sigma\mu} - \partial_\nu \Gamma^\lambda_{\sigma\mu} + \Gamma^\lambda_{\rho\mu} \Gamma^\rho_{\sigma\nu} - \Gamma^\lambda_{\rho\nu} \Gamma^\rho_{\sigma\mu}. \tag{15}$$

With the Riemann tensor, we define another tensorial object of two lower rank by contracting (summing over the indices of) the Riemann tensor:

$$R_{\mu\nu} := R^\lambda_{\mu\lambda\nu}. \tag{16}$$

This new quantity, called the **Ricci tensor**, still contains information about the curvature of your manifold from the Riemann tensor but now in a more compact form. The Ricci tensor therefore also describes curvature of a manifold but in a more compact form.

We then define another quantity, called the **Ricci scalar** (or often referred to as the scalar curvature), as the trace of the Ricci tensor with respect to the inverse metric:

$$R := g^{\mu\nu} R_{\mu\nu}. \tag{17}$$

Since the metric defines the geometry of your manifold near a point and the Ricci tensor is a compact measure for the curvature of the manifold, this Ricci scalar describes the **average curvature at a point**. It should be no surprise that a measure for the average curvature at a point depends on some notion for what the curvature of a manifold looks like and what the geometry around a single point looks like.

(Link to general relativity by covering semi-Riemannian geometry and the Einstein field equations)

(formally define a spacetime)

2.4 Definition of Spacetime

2.5 Einstein Field Equations

(derive the EFEs from varying the action)

The Einstein field equations are defined as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{18}$$

3 Geometry of Hypersurfaces

4 3+1 Decomposition

4.1 Defining lapse function and shift vector

We wish to develop a *general method* for slicing a spacetime manifold $\mathcal{M}$ into spatial hypersurfaces so that their evolution can be tracked over a coordinate time t. We first imagine creating two slices, Σ_t and Σ_{t+dt} and define the normal vector to any surface as

$$n_\mu = -\alpha \nabla_\mu t \tag{19}$$

where α is added as a normalization constant, t is added to correctly order the hypersurfaces in time, ∇_μ is the covariant derivative, added because the normal vector to a plane will be the partial derivatives of the vector or function describing the plane (**need better reasoning here**), and the negative sign is introduced since we are using the metric signature $(-,+,+,+)$. This means we have

$$n_\mu = (-\alpha, 0, 0, 0) \tag{20}$$

and the dot product of the normal vector with itself is

$$n_\mu n^\mu \equiv g^{\mu\nu} n_\mu n_\nu = -1. \tag{21}$$

With two hypersurfaces, Σ_t and Σ_{t+dt}, we would first like to know how the change in coordinate time corresponds to a change in proper time for an observer on the surfaces. (again, really need image here) The proper time separation will be $n^\mu d\tau$ and we want to know how this is related to dt. To do this, we take the dot product of the separation vector $n^\mu d\tau$ and the gradient of coordinate time (describing the rate of change of coordinate time along the direction of $n^\mu d\tau$):

$$dt = (\nabla_\mu t)(n^\mu d\tau). \tag{22}$$

Using Eq. (19), we have

$$dt = \left(-\frac{n_\mu}{\alpha}\right)(n^\mu d\tau) = \frac{d\tau}{\alpha} \implies \boxed{\alpha dt = d\tau}. \tag{23}$$

So we have learned that the normalization constant α of the normal vectors tells us how much proper time τ (measured by an observer on these hypersurfaces) changes as coordinate time advances between the hypersurfaces. For this reason, α is called a **lapse function**.

We would similarly like to know how spatial coordinates change for an observer on the hypersurface as coordinate time advances. First note that we can introduce a factor β^i into the definition of n^μ's spatial components without alerting our normalization condition $n^\mu n_\mu = -1$ (**I don't understand this at all**) so we can say

$$\boxed{n^\mu = \frac{1}{\alpha}(-1, \beta^i)}. \tag{24}$$

To determine how a change in coordinate time corresponds to a change in spatial coordinates on the hypersurface, we again play the same game of taking the dot product of the change in the spatial separation vector $\nabla_\mu x^i$ along the change in the observer's proper time vector $n^\mu d\tau$ (again remember that n^μ gives direction to the change in proper time $d\tau$). This results in

$$dx^i = (\nabla_\mu x^i)(n^\mu d\tau) \tag{25}$$

and we can say that $\nabla_\mu x^i = \partial_\mu x^i$ since x^i are just coordinates (**I don't understand why the Christoffel symbols vanish...**). We can then say that $\partial_\mu x^i = \delta^i_\mu$ (since $\partial_y x = 0$ but $\partial_x x = 1$), leaving us with

$$dx^i = \delta^i_\mu n^\mu d\tau = n^i d\tau = \frac{\beta^i}{\alpha} d\tau \tag{26}$$

then using Eq. (23), we have

$$dx^i = \frac{\beta^i}{\alpha}\alpha dt \implies \boxed{dx^i = \beta^i d\tau}. \tag{27}$$

β therefore measures the rate at which a normal observer would notice the spatial coordinates change (if an observer at Σ_t could see the hypersurface and all of its coordinates evolve to Σ_{t+dt}, they would see a spacetime difference of $(t+dt, x^i - \beta^i dt)$ ... **shaky on this... not totally clear. Not clear on the next part either**). This is why β is called the **shift vector**. We can then define a time vector according to a coordinate observer (one outside of $\mathcal{M}$... the person evolving the simulation) as

$$t^\mu = \alpha n^\mu + \beta^\mu = (1,0,0,0) \tag{28}$$

4.2 Decomposing tensors

We now wish to split tensors into spatial and temporal components. Given a vector A^μ, we want to decompose it into a component aligned with the normal vector (along the direction of proper time flow) and components that are aligned with the spatial slices. We can always decompose a tensor into components that are parallel and perpendicular to n^μ:

$$A^\mu = A^\mu_\perp + A^\mu_\parallel \tag{29}$$

where $A^\mu_\perp$ would be the normal component and $A^\mu_\parallel$ would be aligned with the spatial slice. First define the magnitude of the vector parallel to n^μ as

$$\phi := -n_\nu A^\nu \tag{30}$$

(negative sign added because n_ν is timelike, so for future-pointing vectors, n^ν will be positive). Because the parallel component of A^μ will point in the same direction as n^μ, we have that

$$A^\mu_\parallel = \phi n^\mu = (-n_\nu A^\nu) n^\mu = -n_\nu n^\mu A^\nu. \tag{31}$$

Now to compute $A^\mu_\perp$, we rearrange the definition given in Eq. (29) and apply what we have found in Eq. (31):

$$A^\mu_\perp = A^\mu - A^\mu_\perp = A^\mu + n_\nu n^\mu A^\nu \tag{32}$$

and use the trick $A^\mu = \delta^\mu_\nu A^\nu$ so we have

$$A^\mu_\perp = \delta^\mu_\nu A^\nu + n_\nu n^\mu A^\nu = (\delta^\mu_\nu + n_\nu n^\mu) A^\nu. \tag{33}$$

We have therefore found that, given a spacetime vector A^ν, the operator $\delta^\mu_\nu + n_\nu n^\mu$ will return only the components of A^ν that are aligned with the spatial slices. For this reason, we define a projection operator γ^μ_ν as

$$\gamma^\mu_\nu := \delta^\mu_\nu + n_\nu n^\mu. \tag{34}$$

As an informative example, we will decompose a rank 2 tensor into its spatial and normal components using the method developed above. The object we would like to get spatial projections of is $\nabla_\mu n_\nu$; how the normal vectors change with respect to each direction.

Motivated by our previous derivation, we will use a similar trick by letting

$$\nabla_\mu n_\nu = \delta^\lambda_\mu \delta^\sigma_\lambda \nabla_\lambda n_\sigma \tag{35}$$

and we rearrange Eq. (34) as $\delta^\beta_\alpha = \gamma^\beta_\alpha - n_\alpha n^\beta$, leaving us with

$$\nabla_\mu n_\nu = (\gamma^\lambda_\mu - n_\mu n^\lambda)(\gamma^\sigma_\nu - n_\nu n^\sigma)\nabla_\lambda n_\sigma \tag{36}$$

and you can multiply this out to get an idea of what each component will mean individually. In Thomas Baumgarte's lecture, he eventually defines an acceleration from one of these terms. I will skip this now but may come back to finish it with textbooks.

The following definition is made from one of the terms in the previous expression:

$$\boxed{K_{\mu\nu} := -\gamma^\lambda_\mu \gamma^\sigma_\nu \nabla_\lambda n_\sigma}. \tag{37}$$

This is called the **extrinsic curvature**: it is the spatial projection of the gradient of the normal vector (how the normal vector changes along multiple directions). This captures how each slice Σ_t is curved in its embedding within the manifold $\mathcal{M}$. (So I think intrinsic curvature will be captured by a projection of the metric along all points of the manifold and this extrinsic curvature is how curved the *borders* of Σ_t along $\mathcal{M}$ are).

As you would expect, the spatial covariant derivative of a spatial vector $A^\mu_\perp$ is taken by spatially-projecting the derivative

$$D_\mu A^\nu_\perp := \gamma^\lambda_\mu \gamma^\nu_\sigma \nabla_\lambda A^\mu_\perp. \tag{38}$$

If you want to take spatial covariant derivatives of a spacetime vector, the following operation is used:

$$D_\mu A^\nu_\perp = \gamma^\lambda_\mu \gamma^\nu_\sigma \nabla_\lambda A^\mu_\perp. \tag{39}$$

This covariant derivative would return zero if the vector was parallel transported (meaning if was moved along the curve without changing its direction relative to that curve), meaning $\nabla_\mu A^\nu = 0$ in 4D and $D_\mu A^\nu_\perp = 0$ on the spatial slices.

Lie Derivative: I don't know where its form comes from (how it's derived) but it tells you what part of changes to the input tensor are *not* due to simple coordinate transformations after it is dragged along some curve.

Contracted Biancchi identity:

$$\nabla_\mu G^{\mu\nu} = 0 \tag{40}$$

(divergence of Einstein tensor is zero). Using the field equations, this also tells us that

$$\nabla_\mu T^{\mu\nu} = 0, \tag{41}$$

which is the conservation of energy.

Recipe for decomposing a theory into its spatial and normal (temporal) components:

1. Decompose the theory's variables

    (a) In GR, the fundamental quantity is the metric so we must reduce $g_{\mu\nu}$ into a component which only contains information about spatial curvature and another component that only contains information about time in the theory

2. Decompose the equations (which will give you constraints on the theory's quantities (like momentum and energy)

    (a) In GR, this is the action of decomposing the EFEs

4.3 Decomposing the metric

To decompose the metric, use the same trick as before where we say

$$g_{\mu\nu} = \delta^{\lambda}_{\mu}\delta^{\sigma}_{\nu}g_{\lambda\sigma} \tag{42}$$

and

$$\delta^{\mu}_{\nu} = \gamma^{\mu}_{\nu} - n_{\nu}n^{\mu} \tag{43}$$

so we have

$$\begin{aligned} g_{\mu\nu} &= (\gamma_{\mu}n^{\lambda} - n^{\lambda}_{\mu})(\gamma^{\sigma}_{\nu} - n_{\nu}n^{\sigma})g_{\lambda\sigma} \\ &= \gamma^{\lambda}_{\mu}\gamma^{\sigma}_{\nu}g_{\lambda\sigma} - \gamma_{\mu}n^{\lambda}n_{\nu}n^{\sigma}g_{\lambda\sigma} - \gamma^{\sigma}_{\nu}n^{\lambda}_{\mu}g_{\lambda\sigma} + n_{\mu}n^{\lambda}n_{\nu}n^{\sigma}g_{\lambda\sigma} \end{aligned} \tag{44}$$

and

$$\gamma^{\lambda}_{\mu}n_{\nu}n^{\sigma}g_{\lambda\sigma} = \gamma^{\lambda}_{\nu}n_{\mu}n_{\sigma} \tag{45}$$

and since γ^{λ}_{ν} describes a spatial slice while the n-vectors describe vectors normal to those spatial slices, there will be no components that the two variables can project onto eachother, so the whole sum goes to zero. This is true for both cross-terms in the previous expansion, leaving us with

$$\begin{aligned} g_{\mu\nu} &= \gamma^{\lambda}_{\mu}\gamma^{\sigma}_{\nu}g_{\lambda\sigma} + n_{\mu}n^{\lambda}n_{\nu}n^{\sigma}g_{\lambda\sigma} \\ &= \gamma^{\lambda}_{\mu}\gamma_{\nu\lambda} + n_{\mu}n^{\lambda}n_{\nu}n_{\lambda} \\ &= \gamma_{\mu\nu} - n_{\mu}n_{\nu} \\ \implies & \boxed{\gamma_{\mu\nu} = g_{\mu\nu} + n_{\mu}n_{\nu}} \end{aligned} \tag{46}$$

where $\gamma_{\mu\nu}$ is defined as the <u>induced spatial metric</u>; given a full spacetime metric $g_{\mu\nu}$, one uses Eq. (46) to derive a purely spatial metric on a hypersurface Σ_t.

We can now use this spatial metric to raise and lower indices just as the full spacetime metric can. Given a vector β^{ν},

$$\beta_{\mu} = g_{\mu\nu}\beta^{\nu} = (\gamma_{\mu\nu} - n_{\mu}n_{\nu})\beta^{\nu} \tag{47}$$

and if β^{ν} is fully spatial, $n_{\mu}n_{\nu}\beta^{\nu} = 0$ since they are always orthogonal, meaning

$$\beta^{\mu} = \gamma_{\mu\nu}\beta^{\nu}. \tag{48}$$

Additionally, since we found

$$n_\mu \beta^\mu = 0 \tag{49}$$

and said previously that $n_\mu = (-\alpha, 0, 0, 0)$ (since the normal vectors only point in the direction of forward time), the only non-zero component we have is

$$n_t \beta^t = -\alpha\beta^t = 0 \implies \beta^t = 0, \tag{50}$$

which tells us that any contravariant spatial vector like β^μ must have vanishing normal components. Note that it could be the case that covariant spatial vectors have normal components since

$$n^\mu \beta_\mu = n^t \beta_t + n^i \beta_i = 0. \tag{51}$$

This equality must hold but it does not prevent the scenario where $n^t\beta_t = -n^i\beta_i$.

We then write

$$\begin{aligned} g^{\mu\nu} &= \gamma^{\mu\nu} - n^\mu n^\nu \\ &= \begin{pmatrix} n^t n^t & n^t n^i \\ n^i n^t & n^i n^j \end{pmatrix} = \begin{pmatrix} \alpha^{-2} & -\beta^i \alpha^{-2} \\ -\beta^i\alpha^{-2} & -\alpha^{-2}\gamma^{ij}\beta^i\beta^j \end{pmatrix} \end{aligned} \tag{52}$$

using the fact that $n^\mu = \frac{1}{\alpha}(-1, \beta^i)$. We then invert this matrix to find (I'm trusting the computation done in the lecture here... haven't done it myself):

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_j\beta^j & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix}. \tag{53}$$

We can then define the spacetime interval with the decomposed metric:

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= (-\alpha^2 + \beta_j \beta^j) dt^2 + \beta_i dt dx^i + \beta_i dx^i dt + \gamma_{ij} dx^i dx^j \\ &= -\alpha^2 dt^2 + \gamma_{ij}\beta^i\beta^j dt^2 + 2\gamma_{ij}\beta_j dt dx^i + \gamma_{ij} dx^i dx^j \\ &= -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \end{aligned} \tag{54}$$

So we have successfully decomposed the metric into spatial and normal components, with the induced spatial metric $\gamma_{\mu\nu}$ being derived from a given spacetime metric $g_{\mu\nu}$.

Example 5 (Writing γ_{ij} for Schwarzschild Metric in Isotropic Coordinates)**.** Consider the Schwarzschild line element in isotropic coordinates

$$ds^2 = \left(\frac{1 - \frac{M}{2r}}{1 - \frac{M}{2r}}\right)^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 (dr^2 + r^2 d\Omega). \tag{55}$$

To decompose this metric, since there are no cross-terms ($dtdx^i$), we have that $\beta = 0$ and we have that

$$\alpha = \left(\frac{1 - \frac{M}{2r}}{1 - \frac{M}{2r}}\right) \tag{56}$$

and

$$\gamma_{ij} = \left(1 + \frac{M}{2r}\right)^4 \eta_{ij} \tag{57}$$

by inspection.

5 Decomposing the field equations

Step 2 of decomposing a theory: decomposing the equations. This is what is done below:

We must first decompose the variables that show up in the field equations (Riemann tensor, Ricci tensor, Ricci scalar). **NOTE:** This is the derivation-heavy part and I'm skipping all of the details right now (which *are* important for my purposes and I will eventually go over them). This part will mainly be repreating the results and explaining the steps for applying the decomposition to the Einstein field equations.

First we define the Lie derivative along a normal vector of the spatial metric to be

$$\mathcal{L}_n \gamma_{\mu\nu} = n^\lambda \nabla_\lambda \gamma_{\mu\nu} + \gamma_{\lambda\nu} \nabla_\mu n^\lambda + \gamma_{\mu\lambda} \nabla_\lambda n^\lambda \tag{58}$$

and this somehow (don't understand how yet) leads to

$$\mathcal{L}_n \gamma_{ij} = -2K_{ij}. \tag{59}$$

We also have the pure time derivative of the spatial metric

$$\partial_t \gamma_{ij} = 2 - \alpha K_{ij} + {}^{(3)}\nabla_i \beta_j + {}^{(3)}\nabla_j \beta_i \tag{60}$$

where ${}^{(3)}\nabla_i$ is the purely spatial covariant derivative, defined as

$${}^{(3)}\nabla_i V^j = \partial_i V^j + V^k \, {}^{(3)}\Gamma^j_{ki} \tag{61}$$

where ${}^{(3)}\Gamma^j_{ki}$ are the purely spatial Christoffel symbols, defined exactly as before in Eq. (12) with $g_{\mu\nu}$ being replaced by Γ_{ij} in the expression.

With the three-dimensional Christoffel symbols defined, we can proceed exactly as before when working with the full metric by first defining a purely spatial (three dimensional) Riemann curvature tensor. There are three unique ways to express the spatial Riemann curvature tensor, though, since we can now decompose the full four-dimensional object

1. Twice along the spatial dimensions

2. Twice along the normal direction (flow of coordinate time)

3. Once along the spatial dimensions and once along the normal direction

and each of these three decompositions of the four dimensional Riemann tensor will result in three different expressions, each of which will introduce **constraint equations** on the spatial hypersurfaces when they are used in the field equations. There are only three unique ways to decompose $R_{\mu\nu\lambda\sigma}$ because if we decomposed twice along the spatial dimensions and once along the normal direction, you would always get zero.

Again, I'm only showing results here. I will *need* to work the derivations out for myself once I have the books. From this process, we get:

1. Purely spatial decompositions result in:

$$\gamma_i^\mu \gamma_j^\nu \gamma_k^\lambda \gamma_l^\sigma R_{\mu\nu\lambda\sigma} = {}^{(3)}R_{ijkl} + K_{ij}K_{jl} - K_{il}K_{jk}. \tag{62}$$

   This is called **Gauss' equation**

2. purely normal decompositions result in:

$$\gamma_i^\mu \gamma_j^\nu n^\lambda n^\sigma R_{\mu\nu\lambda\sigma} = \mathcal{L}_n K_{ij} + \frac{1}{\alpha}\,{}^{(3)}\nabla_i\,{}^{(3)}\nabla_j \alpha + K_i^k K_{kj}. \tag{63}$$

   This is called **Ricci's equation**. Notice that the left side only contains derivatives along the normal direction. **there is something wrong with the indices here. figure it out once the books come**

3. Normal-spatial decomposition:

$$\gamma_j^\mu \gamma_j^\nu \gamma_k^\lambda n^\sigma R_{\mu\nu\lambda\sigma} = {}^{(3)}\nabla_j K_{ik} - {}^{(3)}\nabla_i K_{jk} \tag{64}$$

   These are called the **Codazzi-Mainardi equations**

Now with these equations, we have projections of the Riemann curvature tensor and now can build the Ricci tensor and Ricci scalar out of it by contracting each of the three expressions' indices twice (**again, work this out once you have the books**).

After this is done, you have successfully decomposed all of the variables that show up in the field equations so you are set to fully decompose them. Since there are three unique ways to decompose the fundamental quantity, $R_{\mu\nu\lambda\sigma}$, you will have three unique field equations, each of which will result in a constraint on your variables. Carrying out this process results in:

1. Spatial-spatial:

$$\boxed{\partial_t K_{ij} - \alpha(R_{ij} - 2K_{ik}K_j^k + KK_{ij}) = {}^{(3)}\nabla_i\,{}^{(3)}\nabla_j \alpha - 4\pi M_{ij} + \mathcal{L}_\beta K_{ij}} \tag{65}$$

   where M_{ij} are the matter terms, defined as

$$M_{ij} := 2S_{ij} - \gamma_{ij}(S - \rho) \tag{66}$$

   (see below for the definition of ρ) where we define

$$S_{ij} := \gamma_i^\mu \gamma_j^\nu T_{\mu\nu}, \tag{67}$$

   which is the stress observed by a normal observer (one living on the hyperplanes).

2. Normal-normal:

$$\boxed{{}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi\rho} \tag{68}$$

   This called the **Hamiltonian constraint** where

$$\rho := n^\mu n_\nu T_{\mu\nu} \tag{69}$$

   and K^2 is called the mean curvature, defined as $K^2 := (\gamma^{ij}K_{ij})$.

3. Normal-spatial:

$$\boxed{{}^{(3)}\nabla_i(K^{ij} - \gamma^{ij}K) = 8\pi J^i} \tag{70}$$

These are called the **momentum constraint**, where

$$J^i := -\gamma^{i\mu}n^\nu T_{\mu\nu} \tag{71}$$

These three boxed equations along with our previous condition

$$\boxed{\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}} \tag{72}$$

are called the **ADM equations**, which are the full reformulation of the Einstein field equations in the 3+1 decomposition.

6 Geometry of Foliations

7 Scalar-Vector-Tensor-Gravity

Scalar-Vector-Tensor Gravity (SVTG) modifies the action for a gravitational field by adding additional terms to the original action for general relativity, S_{GR};

$$S = S_{\mathrm{GR}} + S_\phi + S_S \tag{73}$$

where S_ϕ is the action of the massive vector field ϕ_μ, defined as

$$S_\phi = -\int d^4x\sqrt{-g}\left[\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - V(\phi_\mu)\right] \tag{74}$$

and S_S is the action for the scalar fields G, μ, and ω, defined as

$$\begin{aligned} S_S = -\int d^4x\sqrt{-g}\Bigg[&\frac{1}{G^3}\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu G\nabla_\nu G - V(G)\right) \\ &+\frac{1}{G}\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\omega\nabla_\nu\omega - V(\omega)\right) + \frac{1}{\mu^2 G}\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\mu\nabla_\nu\mu - V(\mu)\right)\Bigg]. \end{aligned} \tag{75}$$

With this modified action, we derive the field equations in SVTG **(maybe show the steps here)**, which are

$$G_{\mu\nu} + Q_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T'_{\mu\nu} \tag{76}$$

where

$$Q_{\mu\nu} = \left(\nabla^\alpha\nabla_\alpha g_{\mu\nu}\Theta - \nabla_\mu\nabla_\nu\Theta\right)G \tag{77}$$

and $T'_{\mu\nu}$ is the modified stress-energy tensor, which depends on the ordinary matter-momentum tensor of general relativity $T^{\mathrm{GR}}_{\mu\nu}$, the contribution of the ϕ_μ field $T^\phi_{\mu\nu}$, and the scalar G, ω, μ contributions $T^S_{\mu\nu}$:

$$T'_{\mu\nu} = T^{\mathrm{GR}}_{\mu\nu} + T^\phi_{\mu\nu} + T^S_{\mu\nu}. \tag{78}$$

Where

$$T^{\phi}_{\mu\nu} = \omega\Bigg[B^{\alpha}_{i}B_{j\alpha} - \gamma_{ij}\Bigg(\frac{1}{4}B^{\rho\alpha}B_{\rho\alpha} + V(\phi_\mu)\Bigg) + 2\gamma^{\mu}_{i}\gamma^{\nu}_{j}\frac{\partial V(\phi_\mu)}{\partial g^{\mu\nu}}\Bigg]. \tag{79}$$

For conciseness, we break the scalar field contributions into $T^{S}_{\mu\nu} = T^{G}_{\mu\nu} + T^{\omega}_{\mu\nu} + T^{\mu}_{\mu\nu}$, for the introduced G, ω, and ϕ fields. We will explicitly write out the $T^{G}_{\mu\nu}$ term but note that the ω and μ terms take on the same form up to a factor of some constants. Deriving it from the action given in Eq. (75), $T^{G}_{\mu\nu}$ becomes

$$T^{G}_{\mu\nu} = -\frac{1}{G^3}\Bigg[\nabla_\mu G\nabla_\nu G - 2\frac{\partial V(G)}{\partial g^{\mu\nu}} - g_{\mu\nu}\Bigg(\frac{1}{2}\nabla_\alpha G\nabla^\alpha G - V(G)\Bigg)\Bigg]. \tag{80}$$

8 Deriving the ADM Formulation of SVTG

(explain that you need to compute 3 decompositions of each of SVTG's field equations)

First computing the full projection onto our spatial hypersurfaces Σ_t in SVTG, we begin by carrying this out for the theory's tensor field equation. To do this, we note that SVTG's tensor field equation, given in Eq. (76), only adds a $Q_{\mu\nu}$ term and modifies the stress energy tensor, given in Eq. (78). We must therefore additionally decompose both of these terms, which we carry out using Eq. (**cite the following when they're officially written out:** $\gamma^{\mu}_{i}\gamma^{\nu}_{j}g_{\mu\nu} = \gamma_{ij}$ and $\gamma^{\mu}_{i}\nabla_\mu = D_i$):

$$\begin{aligned}\gamma^{\mu}_{i}\gamma^{\nu}_{j}Q_{\mu\nu} &= \gamma^{\mu}_{i}\gamma^{\nu}_{j}\Bigg(\nabla^\alpha\nabla_\alpha g_{\mu\nu}\Theta - \nabla_\mu\nabla_\nu\Theta\Bigg)G \\ &= \Bigg(\nabla^\alpha\nabla_\alpha\gamma^{\mu}_{i}\gamma^{\nu}_{j}g_{\mu\nu}\Theta - \gamma^{\mu}_{i}\nabla_\mu\gamma^{\nu}_{j}\nabla_\nu\Theta\Bigg)G \\ &= \Bigg(\nabla^\alpha\nabla_\alpha\gamma_{ij}\Theta - D_iD_j\Theta\Bigg)G.\end{aligned} \tag{81}$$

We then compute the spatial-spatial projection of the modified stress-energy tensor, beginning with the $T^{\phi}_{\mu\nu}$ term:

$$\begin{aligned}\gamma^{\mu}_{i}\gamma^{\nu}_{j}T^{\phi}_{\mu\nu} &= \omega\gamma^{\mu}_{i}\gamma^{\nu}_{j}\Bigg[B^{\alpha}_{\mu}B_{\nu\alpha} - g_{\mu\nu}\Bigg(\frac{1}{4}B^{\rho\alpha}B_{\rho\alpha} + V(\phi_\mu) + 2\frac{\partial V(\phi_\mu)}{\partial g^{\mu\nu}}\Bigg)\Bigg] \\ &= \omega\Bigg[B^{\alpha}_{i}B_{j\alpha} - \gamma_{ij}\Bigg(\frac{1}{4}B^{\rho\alpha}B_{\rho\alpha} + V(\phi_\mu)\Bigg) + 2\gamma^{\mu}_{i}\gamma^{\nu}_{j}\frac{\partial V(\phi_\mu)}{\partial g^{\mu\nu}}\Bigg].\end{aligned} \tag{82}$$

We then project the scalar contributions to the stress-energy tensor. Since there are three scalar fields G, μ, and ω added to SVTG, we must project each of them. Thankfully, :

$$\gamma^{\mu}_{i}\gamma^{\nu}_{j}T^{G}_{\mu\nu} = -\frac{1}{G^3}\Bigg[D_iGD_jG - 2\gamma^{\mu}_{i}\gamma^{\nu}_{j}\frac{\partial V(G)}{\partial g^{\mu\nu}} - \gamma_{ij}\Bigg(\frac{1}{2}\nabla_\alpha G\nabla^\alpha G - V(G)\Bigg)\Bigg] \tag{83}$$

We now move on to fully projecting the field equations of SVTG along the normal direction. First projecting the added $Q_{\mu\nu}$ term of the tensor field equations by making use of Eq. and Eq. **(again, cite the following facts:)** $n^\mu n^\nu = -1$ and $n^\mu n^\mu g_{\mu\nu} = -1$:

$$\begin{aligned} n^\mu n^\nu Q_{\mu\nu} &= n^\mu n^\nu \Big(\nabla^\alpha \nabla_\alpha g_{\mu\nu}\Theta - \nabla_\mu \nabla_\nu \Theta\Big) G \\ &= -\Big(\nabla^\alpha \nabla_\alpha \Theta + \nabla_\mu n^\mu \nabla_\nu n^\nu \Theta\Big) G. \end{aligned} \tag{84}$$

We must now take a look at what the $\nabla_\mu n^\mu$ terms turn out to be. To do this, we compute

$$\nabla_\mu n^\mu = g^{\mu\sigma}\nabla_\mu n_\sigma = \big(\gamma^{\mu\sigma} - n^\mu n^\sigma\big)\nabla_\mu n_\sigma = \gamma^{\mu\sigma}\nabla_\mu n_\sigma \tag{85}$$

since $n^\mu \nabla_\mu \big(n^\sigma n_\sigma\big) = 0$. To simplify our result to something more familiar, we use the fact that $K := -\gamma^{\mu\sigma}\nabla_\mu n_\sigma = \nabla_\mu n^\mu$ **(cite if you state earlier in paper)**, which allows us to conclude that

$$n^\mu n^\nu Q_{\mu\nu} = -\Big(\nabla^\alpha \nabla_\alpha \Theta + K^2 \Theta\Big) G. \tag{86}$$

Computing the normal-normal projections of the vector stress-energy tensor:

$$n^\mu n^\nu T^\phi_{\mu\nu} = \omega\left[n^\mu n^\nu B^\alpha_\mu B_{\nu\alpha} + \frac{1}{4}B^{\rho\alpha}B_{\rho\alpha} + V(\phi_\mu) + 2\frac{\partial V(\phi_\mu)}{\partial g^{\mu\nu}}\right] \tag{87}$$

Then computing the normal-normal projection of the scalar stress-energy tensor, where we will again only write out the G contribution since the ω and μ contributions will be of the same form:

$$n^\mu n^\nu T^G_{\mu\nu} = -\frac{1}{G^3}\left[K^2 G^2 - 2n^\mu n^\nu \frac{\partial V(G)}{\partial g^{\mu\nu}} + \frac{1}{2}\nabla_\alpha G \nabla^\alpha G - V(G)\right]. \tag{88}$$

We now move on to computing the mixed projection of the added $Q_{\mu\nu}$ term in the SVTG tensor field equations. We compute:

$$\begin{aligned} \gamma^\mu_i n^\nu Q_{\mu\nu} &= \Big(\nabla^\alpha \nabla_\alpha \gamma^\mu_i n^\nu g_{\mu\nu}\Theta - \gamma^\mu_i \nabla_\mu \nabla_\nu n^\nu \Theta\Big) G \\ &= \Big(\nabla^\alpha \nabla_\alpha \gamma^\mu_i n^\nu g_{\mu\nu}\Theta + D_i K \Theta\Big) G. \end{aligned} \tag{89}$$

Further inspecting the $\gamma^\mu_i n^\nu g_{\mu\nu}$ term:

$$\gamma^\mu_i n^\nu g_{\mu\nu} = \gamma^\mu_i n^\nu \gamma_{\mu\nu} - \gamma^\mu_i n_\mu n^\nu n_\nu = \gamma^\mu_i n_\mu = n_i = 0, \tag{90}$$

which makes our mixed projection of $Q_{\mu\nu}$:

$$\gamma^\mu_i n^\nu Q_{\mu\nu} = D_i K \tag{91}$$

since we had defined $\Theta = 1/G(x)$.

Computing the mixed projection of the vector stress-energy tensor:

$$\gamma_i^\mu n^\nu T^\phi_{\mu\nu} = \omega\left(B_i^\alpha n^\nu B_{\nu\alpha} + 2\gamma_i^\mu n^\nu \frac{\partial V(\phi_\mu)}{\partial g^{\mu\nu}}\right). \tag{92}$$

Computing the mixed projection of the scalar stress-energy tensor:

$$\gamma_i^\mu n^\nu T^G_{\mu\nu} = \frac{1}{G^3}\left(D_i G^2 K + 2\gamma_i^\mu n^\nu \frac{\partial V(G)}{\partial g^{\mu\nu}}\right) \tag{93}$$

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A Differential Forms

This is Appendix A.

B Causality in Relativity

Give the formal definitions of casualitiy in relativity. Explain timelike, spacelike, and lightlike intervals.