ADM Formulation of Scalar-Tensor-Vector Gravity

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February 1, 2025

Abstract

General relativity has been an immensely successful theory of gravity, leading to the prediction and discovery of phenomena which has shaped our modern understanding of the universe. General relativity does not exist without challenges, however, due to the existence of singularities that are hypothesized to exist in the center of black holes and at the origin of our universe and challenges that face the theory's cosmological paradigm. In attempting to search for theories of gravity beyond general relativity, writing proposed modified theories in the ADM formalism by applying a 3+1 decomposition to them would allow for a more intuitive understanding of the theories and may aid in their mathematical analysis. In this work, the 3+1 decomposition is applied to the field equations of Scalar-Tensor-Vector Gravity (STVG), writing the theory in the ADM Formulation.

Contents

1 Introduction

General relativity, discovered by Albert Einstein in 1915 [1], has been an incredibly successful theory of gravity with far-reaching implications on the nature of reality that are still in the process of being understood. Given its continued ability to stand up to experimental verification[1], general relativity has become our modern paradigm for understanding the nature of gravity, and revived interest in formally approaching cosmological studies in the early twentieth century, a field which is continuing to experience success.

As is the case with any successful scientific paradigm, however, anomalies have begun to arise in theoretical and observational considerations of the theory. The most significant of which, the existence of singularities at the center of black holes and at the origin of the universe, seems to be so fundamental of a problem that various programs of quantum gravity have attempted to either unify all fundamental interactions of nature, as is done in the context of string theory, or quantize the gravitational field, as is the case in loop quantum gravity. The existence of singularities seems to be so fundamental to the theory of relativity that one cannot hope to solve this anomaly within the confines of the theory.

Another fundamental problem arising from astrophysical observations that conflict with the theoretical predictions made by general relativity is the proposed existence of dark matter. Dark matter was first discovered in 1933 when Fritz Zwicky noticed that the observed rate of rotation of stars around galaxies did not fit the rate of rotation predicted by general relativity. From his and future observations of galactic rotation curves, it was determined that the amount of observable matter in these galaxies was not enough to account for these observed high speeds of rotation given the theoretical predictions made by general relativity. From this discrepancy between observation and the predictions made by general relativity, combined with additional observations that have come to challenge other predictions made by general relativity, physicists began to determine that the addition of matter beyond the standard model of particle physics would solve these discrepancies without sacrificing general relativity. Because this matter is theorized and is currently not observed to interact with our standard model of particle physics, it is inherently directly observable and is thus called *dark* matter.

Because the issue of dark matter arises from a conflict between astrophysical observations and the predictions made by general relativity, dark matter does not appear to be as fundamental of a problem to the theory of relativity as the existence of singularities is. As such, there has been development in the direction of *modifying* the underlying theory of relativity to account for the theoretical discrepancies with observation, which has been done with some success[2]. Although it is currently more favorable to ascribe a particle-nature to dark matter since proposed modified theories of gravity have not yet been able to simultaneously explain *all* observational anomalies, it is still reasonable to

[1]For example, see [2, 3, 4, 5, 6]
[2]For example, see [7, 8]

consider modified gravity as a possible explanation for at least *some* of these anomalies.

To support the development of theories of modified gravity, sophisticated analyses of a given theory's fundamental equations are necessary. One common method used to analyze any general four-dimensional theory is called **3+1 decomposition**, which separates the full four-dimensional theory into spatial and temporal components. This method applied to general relativity has been foundational to developing simulations of relativistic systems, such as the merging of black holes and the system's emission of gravitational waves[3]. One of the most successful theories of modified gravity, called **Scalar-Tensor-Vector (STVG) gravity** [9], commonly called MOdified Gravity (MOG), has not yet had this decomposition applied to its fundamental equations that describe the evolution of relativistic systems. To support the development of modified gravity as a possible alternative to dark matter, we will write the field equations STVG in the ADM formalism, which is a Hamiltonian formulation of a theory of gravity, named after the first to write general relativity in this way, Arowitt, Desner, and Misner.

To write STVG in the ADM formalism, one needs to apply a method called the **3+1 decomposition** to its field equations, which allows the theory's four-dimensional variables to be separated into variables only describing the three-dimensional spatial evolution of gravitational systems and the one-dimensional temporal evolution of these spatial variables.

This thesis proceeds as follows: we will first quickly introduce the relevant tools from the 3+1 decomposition necessary to write STVG in the ADM formalism. Citations to sources with additional details, especially the more technically-minded ones, will be provided throughout. We then demonstrate how general relativity was first written in the ADM formalism, and we close by applying the developed methods to our modified theory of gravity. For the reader interested in the mathematical foundations of general relativity and the necessary differential geometry, a more rigorous treatment is given in Appendix A. Much of the derivations needed to arrive at the material presented in Section 2.2 are very long. All the bloody details for one of the derivations needed for the introduction to the ADM formulation of general relativity are provided in Appendix B. The rest of the details are filled in by cited sources, in particular [10].

2 Background Content

2.1 Tools from 3+1 Decomposition

Given a manifold $\mathcal{M}$ endowed with a metric $g_{\mu\nu}$, we wish to undo the beautiful unification of space and time given to us by general relativity and slice the manifold into spatial hypersurface that evolve in some coordinate time t to recover the full structure of the original manifold. To do this, we first imagine

[3] Here is where I'll cite foundational papers in NR. Maybe Frans Pretorius?

two slices, Σ_t and Σ_{t+dt} and define a normal vector on a slice by considering our usual operator for constructing vectors normal to surfaces, the gradient.

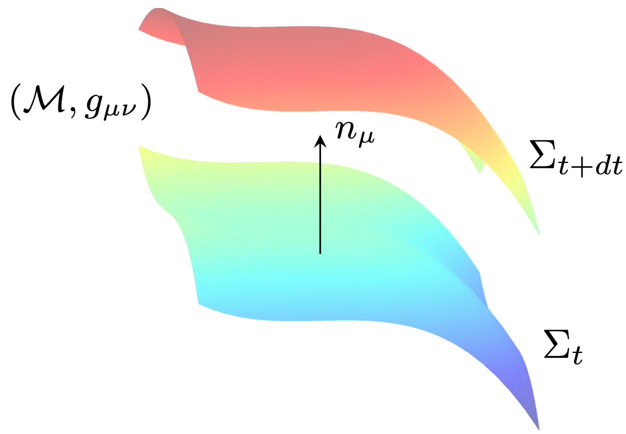

Figure 1: A manifold $\mathcal{M}$, endowed with a metric $g_{\mu\nu}$, is sliced into three-dimensional hypersurfaces Σ_t and Σ_{t+dt}, which are separated by their orientation along the direction of the normal vector n_μ.

Because we want time to increase from surface Σ_t to Σ_{t+dt}, in constructing this normal vector from a gradient, we must also introduce a negative sign to keep it future-pointing. We must additionally add a normalization constant, α, whose exact structure we will analyze soon. This leaves us with the normal vector to a spatial hypersurface defined as

$$n_\mu = -\alpha \nabla_\mu t. \tag{1}$$

Since n_μ is a timelike vector, we normalize it such that

$$n_\mu n_\nu = g^{\mu\nu} n^\nu n_\nu = -1, \tag{2}$$

which reflects our choice of metric signature $(-,+,+,+)$ and resembles the normalization of four-velocity in general relativity.

From this definition of the normal vector, we wish to define the spatial metric induced on the hypersurfaces by $g_{\mu\nu}$. To do this, we must first consider how to decompose a general vector into components parallel and perpendicular to this normal vector. We proceed by first using the fact that any general vector A^μ can be decomposed into components parallel and perpendicular to n^μ, written:

$$A^\mu = A^\mu_\parallel + A^\mu_\perp. \tag{3}$$

Since $A^\mu_\parallel$ is parallel with n^μ, we can write it as having a magnitude of $-n_\nu A^\nu$ and direction of n^μ, where the negative sign has again been added to account for our metric signature. Using this in Eq. (3) and rearranging it to solve for $A^\mu_\perp$ gives us

$$A^\mu_\perp = A^\mu + n_\nu n^\mu A^\nu.$$

We now use the fact that $A^\mu = \delta^\mu_\nu A^\nu$ to express $A^\mu_\perp$ as some operator acting on our vector A^ν:

$$A^\mu_\perp = A^\nu\big(\delta^\mu_\nu + n_\nu n^\mu\big),$$

which we define to be a *projection operator*

$$\gamma^\mu_\nu := \delta^\mu_\nu + n_\nu n^\mu \tag{4}$$

that returns only the spatial components of a given vector A^μ.

We now use this projection operator to define our induced spatial metric. We begin by again using the fact that the Kronceker delta allows us to swap the indices on any tensor:

$$g_{\mu\nu} = \delta^\lambda_\mu \delta^\sigma_\nu g_{\lambda\sigma}.$$

We then rearrange Eq. (4) to write its δ^λ_μ and δ^σ_ν in terms of our newly-derived projection operator:

$$\begin{aligned} g_{\mu\nu} &= (\gamma_\mu n^\lambda - n^\lambda_\mu)(\gamma^\sigma_\nu - n_\nu n^\sigma) g_{\lambda\sigma} \\ &= \gamma^\lambda_\mu \gamma^\sigma_\nu g_{\lambda\sigma} - \gamma_\mu n^\lambda n_\nu n^\sigma g_{\lambda\sigma} - \gamma^\sigma_\nu n^\lambda_\mu g_{\lambda\sigma} + n_\mu n^\lambda n_\nu n^\sigma g_{\lambda\sigma}. \end{aligned}$$

For the two cross-terms in our expansion, we first use the metric to swap indices with one of the normal vectors, which then leaves us with an expression for the inner product between the spatial metric and a normal vector. Since the spatial metric and normal vector are orthogonal, their inner product vanishes, leaving us with

$$g_{\mu\nu} = \gamma^\lambda_\mu \gamma^\sigma_\nu g_{\lambda\sigma} + n_\mu n^\lambda n_\nu n^\sigma g_{\lambda\sigma}. \tag{5}$$

To simplify this further, we again use the metric and our projection operator to swap indices and make use of the normalization condition for the normal vector to compute

$$\begin{aligned} g_{\mu\nu} &= \gamma^\lambda_\mu \gamma_{\nu\lambda} + n_\mu n^\lambda n_\nu n_\lambda \\ &= \gamma_{\mu\nu} - n_\mu n_\nu. \end{aligned}$$

With this, we have found

$$\boxed{\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu}, \tag{6}$$

which is our **induced spatial metric**. Given a spacetime metric $g_{\mu\nu}$, one uses Eq. (6) to derive a purely spatial metric on a hypersurface Σ_t.

Because our spatial hypersurfaces are evolving over coordinate time, we also wish to describe how exactly these slices change over time. In the context of the language we have introduced, we wish to determine the change, captured by the covariant derivative ∇_σ, of our normal vector n_λ and project this quantity onto our spatial surfaces. Because we have two indices that require projections, we will use two projection operators to define the quantity of **extrinsic curvature**,

$$K_{\mu\nu} := -\gamma^\sigma_\mu \gamma^\lambda_\nu \nabla_\sigma n_\lambda.$$

The extrinsic curvature tells you how your normal vector changes over the surface of the spatial slices, which gives us a measure of how the spatial hypersurfaces change over time. We then use Eq. (6) to express the extrinsic curvature as:

$$K_{\mu\nu} = -\big(g^{\sigma}_{\mu} + n_\mu n^\sigma\big)\big(g^{\lambda}_{\nu} + n_\nu n^\lambda\big)\nabla_\sigma n_\lambda$$

and we use the fact that

$$n_\nu n^\lambda \nabla_\sigma n_\lambda = n_\nu \nabla_\sigma\big(n^\lambda n_\lambda\big) = 0 \tag{7}$$

to write our expansion as

$$K_{\mu\nu} = -\big(g^{\sigma}_{\mu} g^{\lambda}_{\nu} + n_\mu n^\sigma g^{\lambda}_{\nu}\big)\nabla_\sigma n_\lambda.$$

We then use our metric terms to swap indices in this expression:

$$\begin{aligned} K_{\mu\nu} &= -g^{\sigma}_{\mu} g^{\lambda}_{\nu}\nabla_\sigma n_\lambda - n_\mu n^\sigma g^{\lambda}_{\nu}\nabla_\sigma n_\lambda \\ &= -g^{\sigma}_{\mu}\nabla_\sigma n_\mu - n_\mu n^\sigma \nabla_\sigma n_\nu. \end{aligned}$$

We then define the acceleration of a normal vector as

$$a_\nu := n^\sigma \nabla_\sigma n_\nu \tag{8}$$

to conclude

$$\boxed{K_{\mu\nu} = -\nabla_\mu n_\nu - n_\mu a_\nu}. \tag{9}$$

Before moving on to applying these tools to the field equations of general relativity, we must express the extrinsic curvature in a more physically intuitive form: as the time derivative of the induced spatial metric. To do this, we must introduce the concept of a **Lie derivative**, which is a derivative that measures differences in a tensor purely due to coordinate transformations, which are related to how the metric changes over time. The formal definition of the Lie derivative for any general tensor is provided in [10], but for the purposes of this paper, we will only cite the result of when the Lie derivative along the normal vector of the induced spatial metric is computed. This results in:

$$\mathcal{L}_n \gamma_{\mu\nu} = n^\lambda \nabla_\lambda \gamma_{\mu\nu} + \gamma_{\lambda\nu}\nabla_\mu n^\lambda + \gamma_{\mu\lambda}\nabla_\lambda n^\lambda.$$

Using our definition of the spatial metric, we then express this as

$$\mathcal{L}_n \gamma_{\mu\nu} = n^\lambda \nabla_\lambda\big(g_{\mu\nu} + n_\mu n_\nu\big) + \big(g_{\lambda\nu} + n_\lambda n_\nu\big)\nabla_\mu n^\lambda + \big(g_{\mu\lambda} + n_\mu n_\lambda\big)\nabla_\lambda n^\lambda.$$

Using the fact that the geometry described by the metric does not change as we move across it, written as

$$\nabla_\sigma g_{\mu\nu} = 0,$$

and the result from Eq. (7) twice, we get

$$\mathcal{L}_n \gamma_{\mu\nu} = n^\lambda \nabla_\lambda\big(n_\mu n_\nu\big) + g_{\lambda\nu}\nabla_\mu n^\lambda + g_{\mu\lambda}\nabla_\nu n^\lambda.$$

Noticing the indices on the remaining terms that contain the metric, we are allowed to pull each of the remaining metrics inside of the covariant derivative and lower the index of the remaining normal vector terms, while using the product rule to expand the term that doesn't depend on the metric:

$$\mathcal{L}_n \gamma_{\mu\nu} = n_\nu n^\lambda \nabla_\lambda n_\mu + n_\mu n^\lambda \nabla_\lambda n_\nu + \nabla_\mu n_\nu + \nabla_\nu n_\mu .$$

Finally, using the definition of the acceleration of the normal vector, Eq. (8), our definition of the extrinsic curvature from Eq. (9), and the fact that n_μ is orthogonal to a_ν, which means they commute and allows us to state $n_\nu a_\mu = n_\mu a_\nu$, we conclude that

$$\boxed{\mathcal{L}_n \gamma_{\mu\nu} = -2K_{\mu\nu}}. \tag{10}$$

This directly relates the extrinsic curvative to how the spatial metric changes over the direction of the normal vector, which captures the direction of time flow in 3+1 decomposition.

With these tools, we are ready to approach decomposing the field equations of general relativity.

2.2 ADM Formulation of General Relativity

The field equations of general relativity are

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{11}$$

where $R_{\mu\nu}$ and R are defined in Appendix A, and Λ is the cosmological constant that incorporates the expansion of the universe plays on the metric $g_{\mu\nu}$. This expansion rate is often set to zero when dealing with "local" applications of general relativity; scales where the expansion rate of the universe does not play a significant effect, such as in black hole dynamics, gravitational waves, and solar system dynamics.

To apply the 3+1 decomposition to the Einstein field equations using the tools introduced in the previous section, we must decompose the Riemann tensor, defined as

$$R^\lambda_{\mu\nu\sigma} = \partial_\nu \Gamma^\lambda_{\mu\sigma} - \partial_\sigma \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\rho\nu} \Gamma^\rho_{\mu\sigma} - \Gamma^\lambda_{\rho\sigma} \Gamma^\rho_{\mu\nu}, \tag{12}$$

since this is the fundamental quantity that appears in the field equations of general relativity.

We must now compute the spatial-spatial projection $\gamma^\mu_i \gamma^\nu_j$, the normal-normal projection $n^\mu n^\nu$, and the mixed projection $\gamma^\mu_i n^\nu$ of the Riemann tensor to begin writing general relativity in the ADM formulation.

The spatial-spatial projection of the Riemann tensor is carried out in Appendix B, which results in:

$${}^{(3)}R^\gamma_{\sigma\alpha\beta} = \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\gamma_\lambda \gamma^\rho_\sigma R^\lambda_{\rho\mu\nu} + K^\gamma_\beta K_{\alpha\sigma} - K^\gamma_\alpha K_{\beta\sigma}, \tag{13}$$

called **Gauss' equation**.

Additionally provided in Appendix B is the contraction of two indices for the spatially-projected Riemann tensor, which results in the spatially-projected Ricci tensor,

$$^{(3)}R_{\sigma\beta} = \gamma^{\nu}_{\beta}\gamma^{\rho}_{\sigma}R_{\rho\nu} + \gamma^{\nu}_{\beta}\gamma^{\rho}_{\sigma}n_{\lambda}n^{\mu}R^{\lambda}_{\rho\mu\nu} + +K^{\alpha}_{\beta}K_{\alpha\sigma} - KK_{\beta\sigma}. \tag{14}$$

This is important because the Ricci tensor appears in the field equations. The Riemann tensor is given more importance because it is more fundamental; one derives the Ricci tensor from the Riemann tensor by contracting two of its indices.

Taking the normal-normal and the mixed projections of the Riemann tensor and then contracting the result to arrive at the projected Ricci tensor follows a process that is very similar to the derivation demonstrated in Appendix B. The results of these derivations will only be quoted in this thesis, but the interested reader can refer to [10] for the full derivation.

When the normal-normal projection of the Riemann tensor is made, we get

$$\gamma^{\mu}_{i}\gamma^{\nu}_{j}n^{\lambda}n^{\sigma}R_{\mu\nu\lambda\sigma} = \mathcal{L}_n K_{ij} + \frac{1}{\alpha}\,^{(3)}\nabla_i\,^{(3)}\nabla_j\alpha + K^{k}_{i}K_{kj}, \tag{15}$$

which is called **Ricci's equation**.

When the mixed projection of the Riemann tensor is made, we get

$$\gamma^{\mu}_{j}\gamma^{\nu}_{j}\gamma^{\lambda}_{k}n^{\sigma}R_{\mu\nu\lambda\sigma} = \,^{(3)}\nabla_j K_{ik} - \,^{(3)}\nabla_i K_{jk}, \tag{16}$$

which is called the **Codazzi-Mainardi equation**.

Having decomposed the fundamental variable of the Einstein field equations, we can now begin inserting our derived expressions for these decomposed variables into Eq. (39). Again, since we had three unique decompositions, we will have three unique expressions of the Einstein field equations.

See [10] for the full process on this substitution and simplification when these steps are carried out. We again only quote the results here. For the spatial-spatial projection of the Ricci tensor, we eventually write the field equations as

$$\partial_t K_{ij} - \alpha(R_{ij} - 2K_{ik}K^{k}_{j} + KK_{ij}) = \,^{(3)}\nabla_i\,^{(3)}\nabla_j\alpha - 4\pi M_{ij} + \mathcal{L}_{\beta}K_{ij}, \tag{17}$$

where M_{ij} are the matter terms, defined as

$$M_{ij} := 2S_{ij} - \gamma_{ij}(S - \rho).$$

We define S_{ij} to just be the spatial projection of the stress-energy tensor;

$$S_{ij} := \gamma^{\mu}_{i}\gamma^{\nu}_{j}T_{\mu\nu}.$$

Intuitively, this would describe all of the spatial stresses and pressures seen by an observer at rest on the manifold at a particular time.

When the normal-normal projected Ricci tensor is inserted into the field equations, and the resulting expression is simplified, one eventually finds that

$$ {}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi\rho, \tag{18} $$

which is called the **Hamiltonian constraint** equation since it constrains the total energy a system may have at a given time. Here, we similarly define ρ to be the normal-normal projection of the stress-energy tensor;

$$ \rho := n^\mu n^\nu T_{\mu\nu}, $$

which represents the energy density that an observer at rest would experience. K^2 is called the mean curvature, defined as $K^2 := K^{ij}K_{ij}$.

When the mixed projection of the Ricci tensor is inserted into the field equations, one eventually finds

$$ {}^{(3)}\nabla_i(K^{ij} - \gamma^{ij}K) = 8\pi J^i, \tag{19} $$

which is called the **momentum constraint** equation, where

$$ J^i := -\gamma^{i\mu} n^\nu T_{\mu\nu} $$

is called the momentum density. Like the energy density, this represents the momentum of an object that an observer at rest would experience.

Equations (17), (18), and (19), along with the following condition:

$$ \partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}, \tag{20} $$

are collectively called the **ADM equations**, which are the full reformulation of the Einstein field equations in the 3+1 decomposition. Equations (17), (18), and (19) capture how systems in a gravitational field evolve, while Eq. (20) captures how the spatial metric changes over time, or how the gravitational field itself changes. Eq. (20) can be derived from our previous relationship between the extrinsic curvature and the Lie derivative, expressed in Eq. (10).

2.3 Scalar-Tensor-Vector-Gravity

Scalar-Tensor-Vector Gravity (STVG) modifies the action for a gravitational field by adding additional terms to the original action for general relativity, S_{GR};

$$ S = S_{\mathrm{GR}} + S_\phi + S_S $$

where S_ϕ is the action of the massive vector field ϕ_μ, defined as

$$ S_\phi = -\int d^4x \sqrt{-g}\left[\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - V(\phi_\mu)\right] $$

and S_S is the action for the scalar fields G, μ, and ω, defined as

$$ \begin{aligned} S_S = -\int d^4x\sqrt{-g}\Bigg[\frac{1}{G^3}\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu G\nabla_\nu G - V(G)\right) \\ +\frac{1}{G}\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\omega\nabla_\nu\omega - V(\omega)\right) + \frac{1}{\mu^2 G}\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\mu\nabla_\nu\mu - V(\mu)\right)\Bigg]. \end{aligned} \tag{21} $$

With this modified action, Moffat derives the tensor field equations of STVG in [9], which are written as

$$G_{\mu\nu} + Q_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T'_{\mu\nu} \tag{22}$$

where

$$Q_{\mu\nu} = \left(\nabla^\alpha \nabla_\alpha g_{\mu\nu} \Theta - \nabla_\mu \nabla_\nu \Theta \right) G$$

and $T'_{\mu\nu}$ is the modified stress-energy tensor, which depends on the ordinary matter-momentum tensor of general relativity $T^{\text{GR}}_{\mu\nu}$, the contribution of the ϕ_μ field $T^{\phi}_{\mu\nu}$, and the scalar G, ω, μ contributions $T^{S}_{\mu\nu}$:

$$T'_{\mu\nu} = T^{\text{GR}}_{\mu\nu} + T^{\phi}_{\mu\nu} + T^{S}_{\mu\nu}. \tag{23}$$

Where

$$T^{\phi}_{\mu\nu} = \omega \left[B^{\alpha}_{i} B_{j\alpha} - \gamma_{ij} \left(\frac{1}{4} B^{\rho\alpha} B_{\rho\alpha} + V(\phi_\mu) \right) + 2\gamma^{\mu}_{i} \gamma^{\nu}_{j} \frac{\partial V(\phi_\mu)}{\partial g^{\mu\nu}} \right]. \tag{24}$$

For conciseness, we break the scalar field contributions into $T^{S}_{\mu\nu} = T^{G}_{\mu\nu} + T^{\omega}_{\mu\nu} + T^{\mu}_{\mu\nu}$, for the introduced G, ω, and ϕ fields. We will explicitly write out the $T^{G}_{\mu\nu}$ term but note that the ω and μ terms take on the same form up to a factor of some constants. Deriving it from the action given in Eq. (21), $T^{G}_{\mu\nu}$ becomes

$$T^{G}_{\mu\nu} = -\frac{1}{G^3} \left[\nabla_\mu G \nabla_\nu G - 2 \frac{\partial V(G)}{\partial g^{\mu\nu}} - g_{\mu\nu} \left(\frac{1}{2} \nabla_\alpha G \nabla^\alpha G - V(G) \right) \right]. \tag{25}$$

3 Results: Deriving the ADM Formulation of SVTG

As explained in the previous section, to write a theory of gravity in the *ADM formulation*, one needs to compute the three possible projections, the spatial-spatial, normal-normal, and mixed projections, for each of the theory's field equations. The only field equation in general relativity is given by Eq. (39) since we only have one fundamental variable, the metric $g_{\mu\nu}$, whose evolution over spacetime dictates gravitational curvature. Since Scalar-Tensor-Vector Gravity modifies the tensor field equation of general relativity, we must compute these projects for each of the added components of the theory in order to write it in the ADM formalism. Since STVG additionally adds four new variables: the vector field ϕ_μ and the scalar fields G, μ, ω, to fully write the theory in the ADM formulation, one would also need to compute the projections of the field equations that result from these, shown in Eqs. (24) and (25). Because only the tensor field equation of STVG will allow us to make a direct comparison with general relativity, decomposing its tensor field equation is of the most importance to us and will be the part we spend the most time discussing. Future work would include additionally computing the projections of STVG's four additional field equations, but for the purposes of this thesis, we will only decompose its tensor field equation.

To compute the projections of STVG's tensor field equation, given in Eq. (22), note that the theory only adds a $Q_{\mu\nu}$ term and modifies the stress energy tensor, given in Eq. (22) in comparison with general relativity's field equations, given in Eq. (39). We must therefore carry out each of our three projections for the $Q_{\mu\nu}$ term and the modified stress-energy tensor $T'_{\mu\nu}$ to decompose the tensor field equations of STVG.

3.1 Spatial-Spatial Projections

First computing the spatial-spatial projection of the added $Q_{\mu\nu}$ term and modified stress-energy tensor in STVG, we begin with the $Q_{\mu\nu}$ term:

$$\gamma_i^\mu \gamma_j^\nu Q_{\mu\nu} = \gamma_i^\mu \gamma_j^\nu \Big(\nabla^\alpha \nabla_\alpha g_{\mu\nu} \Theta - \nabla_\mu \nabla_\nu \Theta \Big) G$$

To compute this projection, we use the spatial metric as a raising and lowering operator on the spacetime metric,

$$\gamma_{ij} = \gamma_i^\mu \gamma_j^\nu g_{\mu\nu}$$

and use the spatial projection operators to turn the spacetime covariant derivatives into just spatial covariant derivative

$$D_i := \gamma_i^\mu \nabla_\mu .$$

With these tools, we find

$$\begin{aligned} &\Big(\nabla^\alpha \nabla_\alpha \gamma_i^\mu \gamma_j^\nu g_{\mu\nu} \Theta - \gamma_i^\mu \nabla_\mu \gamma_j^\nu \nabla_\nu \Theta \Big) G \\ &= \Big(\nabla^\alpha \nabla_\alpha \gamma_{ij} \Theta - D_i D_j \Theta \Big) G = \gamma_i^\mu \gamma_j^\nu Q_{\mu\nu} . \end{aligned} \tag{26}$$

We then compute the spatial-spatial projection of the modified stress-energy tensor, beginning with the contribution of the vector field, $T^\phi_{\mu\nu}$:

$$\begin{aligned} \gamma_i^\mu \gamma_j^\nu T^\phi_{\mu\nu} &= \omega \gamma_i^\mu \gamma_j^\nu \left[B_\mu^\alpha B_{\nu\alpha} - g_{\mu\nu} \left(\frac{1}{4} B^{\rho\alpha} B_{\rho\alpha} + V(\phi_\mu) + 2 \frac{\partial V(\phi_\mu)}{\partial g^{\mu\nu}} \right) \right] \\ &= \omega \left[B_i^\alpha B_{j\alpha} - \gamma_{ij} \left(\frac{1}{4} B^{\rho\alpha} B_{\rho\alpha} + V(\phi_\mu) \right) + 2 \gamma_i^\mu \gamma_j^\nu \frac{\partial V(\phi_\mu)}{\partial g^{\mu\nu}} \right] . \end{aligned} \tag{27}$$

We then project the scalar contributions to the stress-energy tensor. Since there are three scalar fields G, μ, and ω added to STVG, we must project each of them. Thankfully, these terms don't need any additional computations beyond the basic identities used previously. We compute:

$$\gamma_i^\mu \gamma_j^\nu T^G_{\mu\nu} = -\frac{1}{G^3} \left[D_i G D_j G - 2 \gamma_i^\mu \gamma_j^\nu \frac{\partial V(G)}{\partial g^{\mu\nu}} - \gamma_{ij} \left(\frac{1}{2} \nabla_\alpha G \nabla^\alpha G - V(G) \right) \right] \tag{28}$$

3.2 Normal-Normal Projections

We now move on to fully projecting the field equations of STVG along the normal direction. First projecting the added $Q_{\mu\nu}$ term of the tensor field equations by making use of our normal vectors' normalization condition, given in Eq. (2), we compute

$$\begin{aligned} n^\mu n^\nu Q_{\mu\nu} &= n^\mu n^\nu \Big(\nabla^\alpha \nabla_\alpha g_{\mu\nu} \Theta - \nabla_\mu \nabla_\nu \Theta \Big) G \\ &= -\Big(\nabla^\alpha \nabla_\alpha \Theta + \nabla_\mu n^\mu \nabla_\nu n^\nu \Theta \Big) G. \end{aligned}$$

We must now take a look at what the $\nabla_\mu n^\mu$ terms turn out to be. To do this, we compute

$$\nabla_\mu n^\mu = g^{\mu\sigma} \nabla_\mu n_\sigma = \big(\gamma^{\mu\sigma} - n^\mu n^\sigma \big) \nabla_\mu n_\sigma = \gamma^{\mu\sigma} \nabla_\mu n_\sigma$$

since $n^\mu \nabla_\mu \big(n^\sigma n_\sigma \big) = 0$. To simplify our result to something more familiar, we use the fact that $K := -\gamma^{\mu\sigma} \nabla_\mu n_\sigma = \nabla_\mu n^\mu$, which allows us to conclude that

$$n^\mu n^\nu Q_{\mu\nu} = -\Big(\nabla^\alpha \nabla_\alpha \Theta + K^2 \Theta \Big) G. \tag{29}$$

Computing the normal-normal projections of the vector stress-energy tensor:

$$n^\mu n^\nu T^\phi_{\mu\nu} = \omega \Big[n^\mu n^\nu B^\alpha_\mu B_{\nu\alpha} + \frac{1}{4} B^{\rho\alpha} B_{\rho\alpha} + V(\phi_\mu) + 2 \frac{\partial V(\phi_\mu)}{\partial g^{\mu\nu}} \Big] \tag{30}$$

Then computing the normal-normal projection of the scalar stress-energy tensor, where we will again only write out the G contribution since the ω and μ contributions will be of the same form:

$$n^\mu n^\nu T^G_{\mu\nu} = -\frac{1}{G^3} \Big[K^2 G^2 - 2 n^\mu n^\nu \frac{\partial V(G)}{\partial g^{\mu\nu}} + \frac{1}{2} \nabla_\alpha G \nabla^\alpha G - V(G) \Big]. \tag{31}$$

3.3 Mixed Projections

We now move on to computing the mixed projection of the added $Q_{\mu\nu}$ term in the STVG tensor field equations. We compute:

$$\begin{aligned} \gamma^\mu_i n^\nu Q_{\mu\nu} &= \Big(\nabla^\alpha \nabla_\alpha \gamma^\mu_i n^\nu g_{\mu\nu} \Theta - \gamma^\mu_i \nabla_\mu \nabla_\nu n^\nu \Theta \Big) G \\ &= \Big(\nabla^\alpha \nabla_\alpha \gamma^\mu_i n^\nu g_{\mu\nu} \Theta + D_i K \Theta \Big) G. \end{aligned}$$

Further inspecting the $\gamma^\mu_i n^\nu g_{\mu\nu}$ term:

$$\gamma^\mu_i n^\nu g_{\mu\nu} = \gamma^\mu_i n^\nu \gamma_{\mu\nu} - \gamma^\mu_i n_\mu n^\nu n_\nu = \gamma^\mu_i n_\mu = n_i = 0,$$

which makes our mixed projection of $Q_{\mu\nu}$:

$$\gamma^\mu_i n^\nu Q_{\mu\nu} = D_i K \tag{32}$$

since we had defined $\Theta = 1/G(x)$.

Computing the mixed projection of the vector stress-energy tensor:

$$\gamma_i^\mu n^\nu T^\phi_{\mu\nu} = \omega\left(B_i^\alpha n^\nu B_{\nu\alpha} + 2\gamma_i^\mu n^\nu \frac{\partial V(\phi_\mu)}{\partial g^{\mu\nu}}\right). \tag{33}$$

Computing the mixed projection of the scalar stress-energy tensor:

$$\gamma_i^\mu n^\nu T^G_{\mu\nu} = \frac{1}{G^3}\left(D_i G^2 K + 2\gamma_i^\mu n^\nu \frac{\partial V(G)}{\partial g^{\mu\nu}}\right) \tag{34}$$

The collection of equations (26), (29), and (32) with the corresponding decompositions of the vector stress energy tensor given in Equations (27), (30), and (33), and the corresponding decompositions of the scalar stress energy tensor given in Equations (28), (31), and (34), give us the full decomposition of STVG's tensor field equations. These are the only modifications that have a direct analog to the field equations of general relativity, since there aren't any separate vector or scalar field equations in general relativity as there are in STVG. Because of this, with this collection of equations, we have written Scalar-Tensor-Vector Gravity in the ADM formulation.

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A Differential Geometry and General Relativity

A.1 Notion of a Manifold

The framework of general relativity is written in the language of differential geometry, so providing a short outline of its relevant elements is necessary. We first define the topological space that general relativity is written on.

Definition 1. A **manifold** is a topological space $\mathcal{M}$ such that each point $p \in \mathcal{M}$ has an open neighborhood U which is homeomorphic to an open subset of a Euclidean space, $\mathbb{R}^n$.

It is natural to formulate general relativity on manifolds because this space allows us to define notions of distances, angles, and curvature. Manifolds also provide the flexibility to describe the region near each point as flat, even when the global topology is not necessarily flat, which is an assumption that, as we will see, plays a crucial role in general relativity. A visualization of a manifold is given in Fig. 2.

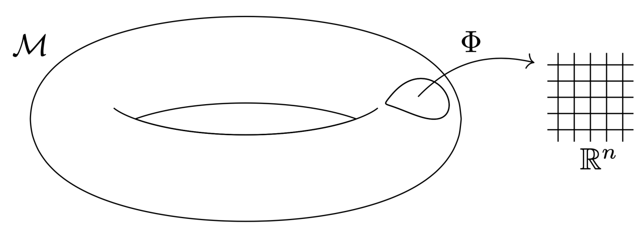

Figure 2: Space local to a point on a manifold will always appear Euclidean.

A.2 Vectors and Tensors on Manifolds

In special relativity, the globally flat geometry naturally defines a four-dimensional vector space structure. When moving to more general topologies, however, this vector space structure is lost. To retrieve these powerful tools in curved geometries, we recover vectors on manifolds in the limit of infinitesimal displacements about a point, which are called **tangent vectors**.

Definition 2. On a manifold $\mathcal{M}$, let $\mathscr{F}$ denote the collection of continuously differentiable functions from $\mathcal{M}$ into $\mathbb{R}^n$. The **tangent vector** V at a point $p \in \mathcal{M}$ is defined to be a map $V : \mathscr{F} \to \mathbb{R}^n$ which is (1) linear and (2) obeys the Leibniz rule:

1. $V(af + bg) = aV(f) + bV(g)$, for all $f, g \in \mathscr{F}$, where $a, b \in \mathbb{R}^n$;
2. $V(fg) = f(p)V(g) + g(p)V(f)$.

The collection of vectors tangent to a point $p \in \mathcal{M}$ is the tangent space $T_p(\mathcal{M})$. The tangent space at a point on a general manifold is illustrated in Fig 3.

In $\mathbb{R}^n$, vectors $V = (V^1, \dots, V^n)$ define the directional derivatives $V^\mu\,(\partial/\partial x^\mu)$ and vice versa[4], so it is natural to express tangent spaces in terms of the set of directional derivatives (∂_μ).

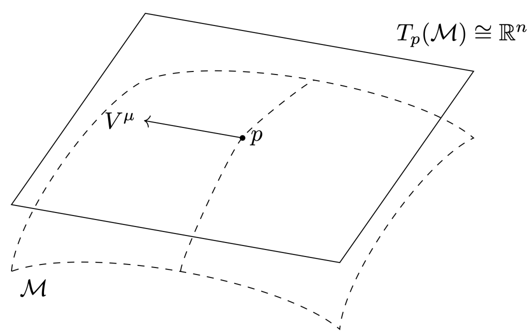

Figure 3: A vector V^μ drawn in the tangent space $T_p(\mathcal{M})$.

With tangent spaces allowing us to describe the set of all possible directions one can move locally on a manifold, we now must introduce the tools for more precisely describing how these directions are measured. We first define linear forms.

Definition 3. A **linear form** at each point $p \in \mathcal{M}$ is the linear mapping $\boldsymbol{\omega}$ such that

$$\boldsymbol{\omega} : T_p(\mathcal{M}) \longrightarrow \mathbb{R}$$

for all $p \in T_p(\mathcal{M})$.

The set of linear forms at p constitutes its own vector space, which is called the ***dual space*** of $T_p(\mathcal{M})$, denoted by $T_p^*(\mathcal{M})$. Vectors defined in the dual space are called ***contravariant***.

Given the natural basis (∂_μ) of $T_p(\mathcal{M})$ associated with a set of coordinates (x^μ) on a manifold, there exists a unique basis for $T_p^*(\mathcal{M})$, denoted by $(\boldsymbol{d}x^\mu)$ such that

$$\langle \boldsymbol{d}x^\mu, \partial_\nu \rangle = \delta^\mu_\nu,$$

where δ^μ_ν, called the Kroceker-delta, is defined as $\delta^\mu_\nu = 0$ when $\nu \neq \mu$ and $\delta^\mu_\nu = 1$ when $\nu = \mu$. This basis $(\boldsymbol{d}x^\mu)$ is called the dual basis of (∂_μ). Note that the natural bases are not the only possible choice in the vector space $T_p(\mathcal{M})$. One

[4]Note that we are using the Einstein summation convention throughout, where a repeated index denotes summation over all its possible values; $V^\mu \frac{\partial}{\partial x^\mu} \equiv \sum_\mu V^\mu \frac{\partial}{\partial x^\mu}$ where $\mu \in \{t, x, y, z\}$ in relativity.

could use a general basis $(\boldsymbol{e}_\mu)$ that is not related to any coordinate system on $\mathcal{M}$. Given $(\boldsymbol{e}_\mu)$, there then exists unique dual bases $(\boldsymbol{e}^\mu) \in T^*(\mathcal{M})$ such that

$$\langle \boldsymbol{e}^\mu, \boldsymbol{e}_\nu \rangle = \delta^\mu_\nu.$$

A smooth assignment of linear forms to each tangent space at points on a manifold is called a ***1-form***. Given a smooth scalar field f on $\mathcal{M}$, the canonical 1-form associated with it is the gradient of f, defined as

$$\nabla f = \frac{\partial f}{\partial x^\mu} \boldsymbol{d}x^\mu.$$

With our definitions of tangent spaces and linear forms, we are now ready to define general tensors, which are the fundamental objects in general relativity. Tensors are a generalization of vectors and linear forms.

Definition 4. At a point $p \in \mathcal{M}$, a **tensor** of type (k, ℓ) with $(k, \ell) \in \mathbb{N}^2$ is a mapping

$$\boldsymbol{T} : \underbrace{T^*_p(\mathcal{M}) \times \cdots \times T^*_p(\mathcal{M})}_{k \text{ times}} \times \underbrace{T_p(\mathcal{M}) \times \cdots \times T_p(\mathcal{M})}_{\ell \text{ times}} \longrightarrow \mathbb{R} \tag{35}$$

that is linear with respect to each of its arguments. The natural number $k + \ell$ is called the rank of the tensor.

Given a basis $(\boldsymbol{e}_\mu)$ of $T_p(\mathcal{M})$ and the corresponding dual basis $(\boldsymbol{e}^\mu) \in T^*_p(\mathcal{M})$, we can expand any tensor $\boldsymbol{T}$ of type (k, ℓ) as

$$\boldsymbol{T} = \boldsymbol{T}^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_\ell} \boldsymbol{e}_{\mu_1} \otimes \cdots \otimes \boldsymbol{e}_{\mu_k} \otimes \boldsymbol{e}^{\nu_1} \otimes \cdots \otimes \boldsymbol{e}^{\nu_\ell}.$$

where the $\otimes$ operation is the tensor product, with components $X^\mu V^\nu$ for vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$.

With tangent spaces and general tensors defined, we can now move on to the key structure of a manifold, the ***metric*** $\boldsymbol{g}$.

Definition 5. Definition of Metric A metric is a (0,2) tensor that allows one to define distances and angles on a given manifold. A metric acts on vectors in the tangent space $T_p(\mathcal{M})$ by

$$\boldsymbol{g} : T_p(\mathcal{M}) \times T_p(\mathcal{M}) \longrightarrow \mathbb{R},$$

is symmetric; $\boldsymbol{g}(u, v) = \boldsymbol{v}, \boldsymbol{u}$, and is non-degenerate; there exists a non-zero $\boldsymbol{v} \in T_p(\mathcal{M})$ such that $\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = 0$ for all $\boldsymbol{u} \in T_p(\mathcal{M})$.

In a given basis $(\boldsymbol{e}_\mu)$ of $T_p(\mathcal{M})$, the components of $\boldsymbol{g}$ are given by the matrix $\boldsymbol{g}_{\mu\nu}$ defined by the expression for a general tensor given in Eq. (35) with $(k, \ell) = (0, 2)$:

$$\boldsymbol{g} = g_{\mu\nu} \boldsymbol{e}^\mu \otimes \boldsymbol{e}^\nu.$$

For any vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, we then have that

$$\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = g_{\mu\nu} u^\mu v^\nu.$$

A.3 Connections on Manifolds and Curvature

In relativity and related theories, the space of physical events is represented by a **Lorentzian manifold** $\mathcal{M}$. Unlike a Riemannian manifold, which requires the metric to be positive definite, a Lorentzian manifold allows the metric to have an indefinite signature, meaning inner products can be positive, negative, or zero. This feature preserves the causal structure of spacetime by distinguishing between timelike, spacelike, and lightlike intervals[5]. Additionally, a Lorentzian manifold has a tangent space at each point that is locally homeomorphic to flat spacetime, described by the Minkowski metric $\eta_{\mu\nu}$. The tangent space represents the set of possible directions for movement tangential to a point, allowing the definition of differential objects like gradients on the manifold. The local flatness of the Lorentzian manifold at each point ensures that, on small scales, spacetime appears flat, preserving the validity of special relativity locally.

The manifold is endowed with a metric $g_{\mu\nu}$, which smoothly assigns a value to each point p on $\mathcal{M}$. This smoothness is crucial for ensuring the continuity and differentiability of the manifold. The metric allows the calculation of lengths and angles on $T_p\mathcal{M}$, the tangent space at each point. The metric $g_{\mu\nu}$ encodes the local geometry of the manifold near a point.

The tangent space at a point provides a way to approximate the manifold by flat spacetime at that point. Specifically, at any point p on the manifold, the tangent space $T_p\mathcal{M}$ locally resembles flat spacetime:

$$(\mathcal{M}, g_{\mu\nu}) \xrightarrow{T_p(\mathcal{M})} (\mathbb{R}^{1,n}, \eta_{\mu\nu}).$$

Every Lorentzian manifold comes with a **Levi-Civita connection**, which is a specific type of affine connection that is associated with a Lorentzian manifold. An **affine connection** is a geometrical object that connects tangent spaces at points on a manifold so that derivatives can be taken on the manifold.

[5]See Appendix B for an explanation of causality in relativity.

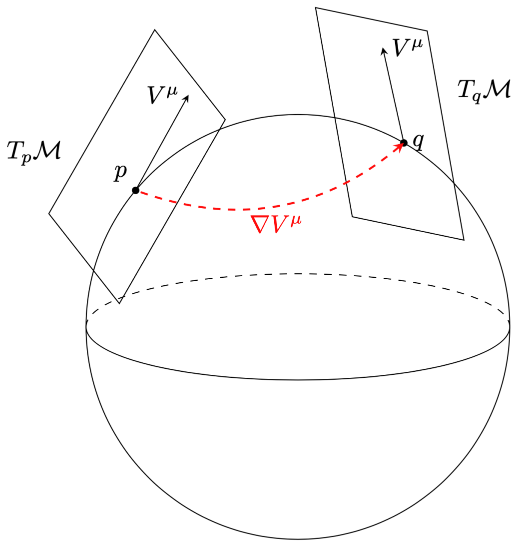

Figure 4: The affine connection from point p to q on a spherical manifold. The tangent spaces and a vector in them for each point are also shown.

This affine connection (just referred to as *connection*) is used to define the covariant derivative of a vector V^μ, defined as

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\lambda} V^\lambda$$

where the $\partial_\nu V^\mu$ term captures how the vector itself changes from p to q (as usual) and the $\Gamma^\mu_{\nu\lambda} V^\lambda$ term, called **Christoffel symbols**, captures how the shape of the manifold (V^μ's basis vectors) changes along the path. The Christoffel symbols are the coefficients of the connection. Since the Christoffel symbols capture how the manifold changes from one point to another, it shouldn't be surprising that their definition contains first derivatives of the metric;

$$.\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\sigma\mu} \left(\frac{\partial g_{\sigma\nu}}{\partial x^\lambda} + \frac{\partial g_{\sigma\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\sigma} \right). \tag{36}$$

In a flat spacetime, the metric would never change from one point to another, meaning $\Gamma^\mu_{\nu\lambda} V^\lambda = 0$, which means our covariant derivative would become

$$\nabla_\nu V^\mu = \partial_\nu V^\mu,$$

which is the regular definition of the gradient operator (so the covariant derivative is just the generalization of the gradient operator to curved spacetimes). If the covariant derivative of a vector along a path is zero, this means that the vector has not changed its orientation over the path. In this case, we say it has been *parallel transported* across the manifold.

Note that the shortest path on a manifold, called its **geodesic**, is described by the geodesic equation:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$

where the x^μ terms are the coordinates on the manifold and τ is the proper time; the time measured by the observer moving along the geodesic.

To measure the **curvature of the manifold**, we must know how the metric, which, again, describes how the geometry near a point changes, itself changes over the manifold. Specifically, we must know how the geometry near each point changes and how these changes vary from point to point. The curvature of the manifold, which encapsulates how the manifold deviates from flatness, is described by the **Riemann curvature tensor**, defined as

$$R^\lambda_{\sigma\mu\nu} := \partial_\mu\Gamma^\lambda_{\sigma\mu} - \partial_\nu\Gamma^\lambda_{\sigma\mu} + \Gamma^\lambda_{\rho\mu}\Gamma^\rho_{\sigma\nu} - \Gamma^\lambda_{\rho\nu}\Gamma^\rho_{\sigma\mu}. \tag{37}$$

With the Riemann tensor, we define another tensorial object of two lower rank by contracting (summing over the indices of) the Riemann tensor:

$$R_{\mu\nu} := R^\lambda_{\mu\lambda\nu}.$$

This new quantity, called the **Ricci tensor**, still contains information about the curvature of your manifold from the Riemann tensor but now in a more compact form. The Ricci tensor therefore also describes curvature of a manifold but in a more compact form.

We then define another quantity, called the **Ricci scalar** (or often referred to as the scalar curvature), as the trace of the Ricci tensor with respect to the inverse metric:

$$R := g^{\mu\nu}R_{\mu\nu}.$$

Since the metric defines the geometry of your manifold near a point and the Ricci tensor is a compact measure for the curvature of the manifold, this Ricci scalar describes the **average curvature at a point**. It should be no surprise that a measure for the average curvature at a point depends on some notion for what the curvature of a manifold looks like and what the geometry around a single point looks like.

With these tools in place, we may now approach the cornerstone of general relativity, the Einstein field equations. The Riemann tensor, defined in Eq. (37), is the fundamental variable to the field equations. The Ricci tensor and Ricci scalar are the only variables that actually appear in the field equations, but they are both arrived at from the Riemann tensor, so they aren't treated as being the fundamental quantity. There are countless resources to find a proper derivation of the Einstein field equations, which is arrived at by constructing the proper action and applying the principle of least action to it. For the purposes of this thesis, I will only provide the correct action, called the Einstein-Hilbert action, which leads to the field equations of general relativity when the principle of least action, $\delta S = 0$, is applied to it. The Einstein-Hilbert action is

$$S = \frac{c^4}{4\pi G} \int R\sqrt{-g}\ d^4x. \tag{38}$$

When the principle of least action is applied to Eq. (38), the Einstein field equations are arrived at, written as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{39}$$

For a full derivation of the Einstein field equations from the Einstein-Hilbert action, see [11].

B Spatial Projection of Riemann Curvature Tensor

For reasons outside the scope of this thesis, we can write the Riemann curvature tensor as a commutator of covariant derivatives:

$$(D_\alpha D_\beta - D_\beta D_\alpha)V^\gamma = {}^{(3)}R^\gamma_{\mu\alpha\beta}V^\mu. \tag{40}$$

We want to now write these covariant derivatives, $D_\alpha D_\beta V^\gamma$, only in terms of spatial components in order to spatially project the Riemann tensor:

$$D_\alpha(D_\beta V^\gamma) = \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\gamma_\rho \nabla_\mu (D_\nu V^\rho)$$

where γ^μ_α transforms the derivative D_β and the $\gamma^\nu_\beta \gamma^\gamma_\rho$ term transforms the rank-two object $D_\beta V^\gamma$. We must now consider what the $D_\nu V^\rho$ term is before proceeding:

$$D_\alpha D_\beta V^\gamma = \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\gamma_\rho \nabla_\mu (D_\nu V^\rho) = \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\gamma_\rho (\nabla_\mu (\gamma^\sigma_\nu \gamma^\rho_\lambda \nabla_\sigma V^\lambda))$$

$$D_\alpha D_\beta V^\gamma = \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\gamma_\rho \Big((\nabla_\mu \gamma^\sigma_\nu)\gamma^\rho_\lambda \nabla_\sigma V^\lambda + \gamma^\sigma_\nu (\nabla_\mu \gamma^\rho_\lambda)\nabla_\sigma V^\lambda + \gamma^\sigma_\nu \gamma^\rho_\lambda \nabla_\mu (\nabla_\sigma V^\lambda)\Big)$$

using the fact that

$$\nabla_\mu \gamma^\sigma_\nu = n^\sigma \nabla_\mu n_\nu + n_\mu \nabla_\mu n^\sigma,$$

we have

$$D_\alpha D_\beta V^\gamma = \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\gamma_\rho \Big(\gamma^\rho_\lambda n^\sigma (\nabla_\mu n_\nu)\nabla_\sigma V^\lambda + \gamma^\rho_\lambda n_\nu (\nabla_\mu n^\sigma)\nabla_\sigma V^\lambda$$

$$+ \nabla_\sigma V^\lambda n_\lambda \gamma^\sigma_\nu (\nabla_\nu n^\rho) + \nabla_\sigma V^\lambda n^\rho \gamma^\sigma_\nu (\nabla_\nu n_\lambda) + \gamma^\sigma_\nu \gamma^\rho_\lambda \nabla_\nu (\nabla_\sigma V^\lambda)\Big)$$

using the fact that $\gamma^\rho_\nu \gamma^\nu_\lambda = \gamma^\rho_\lambda$ and $\gamma^\nu_\lambda n_\nu = 0$, we can show that two terms go to zero by:

$$\gamma^\rho_\lambda n_\nu (\nabla_\mu n^\sigma)\nabla_\sigma V^\lambda = \gamma^\rho_\nu \gamma^\nu_\lambda n_\nu (\nabla_\mu n^\sigma)\nabla_\sigma V^\lambda = 0$$

and

$$\nabla_\sigma V^\lambda n^\rho \gamma^\sigma_\nu (\nabla_\nu n_\lambda) = \nabla_\sigma V^\lambda n^\rho \gamma^\sigma_\rho \gamma^\rho_\nu (\nabla_\nu n_\lambda) = 0$$

This results in

$$D_\alpha D_\beta V^\gamma = \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\gamma_\rho \Bigg(\gamma^\rho_\lambda n^\sigma (\nabla_\mu n_\nu) \nabla_\sigma V^\lambda$$
$$+ \nabla_\sigma V^\lambda n_\lambda \gamma^\sigma_\nu (\nabla_\nu n^\rho) + \gamma^\sigma_\nu \gamma^\rho_\lambda \nabla_\nu (\nabla_\sigma V^\lambda) \Bigg)$$

Distributing the gamma terms and using the fact that $(\nabla_\sigma V^\lambda) n_\lambda = -V^\lambda \nabla_\sigma n_\lambda$, we get

$$D_\alpha D_\beta V^\gamma = \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\gamma_\rho \gamma^\rho_\lambda n^\sigma (\nabla_\mu n_\nu) \nabla_\sigma V^\lambda$$
$$- \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\gamma_\rho V^\lambda \nabla_\sigma n_\lambda \gamma^\sigma_\nu (\nabla_\nu n^\rho) + \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\gamma_\rho \gamma^\sigma_\nu \gamma^\rho_\lambda \nabla_\nu (\nabla_\sigma V^\lambda)$$

After simplifying the γ's now in front of each term and using the definition of the extrinsic curvature $K_{\alpha\beta} = -\gamma^\mu_\alpha \nabla_\mu n_\beta = -\gamma^\mu_\alpha \gamma^\nu_\beta \nabla_\mu n_\nu$, we have

$$D_\alpha D_\beta V^\gamma = -K_{\alpha\beta} \gamma^\gamma_\lambda n^\sigma \nabla_\sigma V^\lambda - K^\gamma_\alpha K_{\beta\lambda} V^\lambda + \gamma^\mu_\alpha \gamma^\sigma_\beta \gamma^\gamma_\lambda \nabla_\mu \nabla_\sigma V^\lambda$$

Now if we consider $D_\beta D_\alpha V^\gamma$:

$$D_\beta D_\alpha V^\gamma = -K_{\beta\alpha} \gamma^\gamma_\lambda n^\sigma \nabla_\sigma V^\lambda - K^\gamma_\beta K_{\alpha\lambda} V^\lambda + \gamma^\mu_\beta \gamma^\sigma_\alpha \gamma^\gamma_\lambda \nabla_\mu \nabla_\sigma V^\lambda$$

We can simplify this expression by swapping the σ and μ indices in the last term and by using the fact that the extrinsic curvature is symmetric ($K_{\mu\nu} = K_{\nu\mu}$), leaving us with

$$D_\beta D_\alpha V^\gamma = -K_{\alpha\beta} \gamma^\gamma_\lambda n^\sigma \nabla_\sigma V^\lambda - K^\gamma_\beta K_{\alpha\lambda} V^\lambda + \gamma^\sigma_\beta \gamma^\mu_\alpha \gamma^\gamma_\lambda \nabla_\sigma \nabla_\mu V^\lambda$$

We can take the difference of these two expressions for the Ricci identity, leaving us with

$$(D_\alpha D_\beta - D_\beta D_\alpha) V^\gamma = (K_{\alpha\lambda} K^\gamma_\beta - K_{\beta\lambda} K^\gamma_\alpha) V^\lambda + \gamma^\mu_\alpha \gamma^\sigma_\beta \gamma^\gamma_\lambda (\nabla_\mu \nabla_\sigma V^\lambda - \nabla_\sigma \nabla_\mu V^\lambda)$$

and since

$$R^\lambda_{\rho\mu\sigma} V^\rho := \nabla_\mu \nabla_\sigma V^\lambda - \nabla_\sigma \nabla_\mu V^\lambda,$$

we have

$$\begin{aligned}(D_\alpha D_\beta - D_\beta D_\alpha) V^\gamma &= (K_{\alpha\lambda} K^\gamma_\beta - K_{\beta\lambda} K^\gamma_\alpha) V^\lambda + \gamma^\mu_\alpha \gamma^\sigma_\beta \gamma^\gamma_\lambda R^\lambda_{\rho\mu\sigma} V^\rho \\ &= {}^{(3)}R^\gamma_{\rho\alpha\beta} V^\rho\end{aligned}$$

from using the three and four dimensional Ricci identities. We then use the fact that $V^\rho = \gamma^\rho_\sigma V^\sigma$ to result in

$${}^{(3)}R^\gamma_{\rho\alpha\beta} V^\sigma = \Bigg(K_{\alpha\rho} K^\gamma_\beta - K_{\beta\rho} K^\gamma_\alpha + \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\gamma_\lambda R^\lambda_{\rho\mu\nu} \Bigg) \gamma^\rho_\sigma V^\sigma$$

so because we have shown that this property hold for any vector V^σ, we have derived

$$\boxed{{}^{(3)}R^\gamma_{\sigma\alpha\beta} = \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\gamma_\lambda \gamma^\rho_\sigma R^\lambda_{\rho\mu\nu} + K^\gamma_\beta K_{\alpha\sigma} - K^\gamma_\alpha K_{\beta\sigma}}, \tag{41}$$

which is called Gauss' equation. This is the fully spatial projection of the 4-dimensional Riemann tensor.

To find the contracted Gauss' equation, which will result in the spatial-spatial projection of the Ricci tensor, we contract over the α and γ indices:

$$^{(3)}R_{\sigma\beta} = \gamma^{\mu}_{\alpha}\gamma^{\nu}_{\beta}\gamma^{\alpha}_{\lambda}\gamma^{\rho}_{\sigma}R^{\alpha}_{\rho\mu\nu} + K^{\alpha}_{\beta}K_{\alpha\sigma} - KK_{\beta\sigma}.$$

Using the fact that $\gamma^{\mu}_{\alpha}\gamma^{\alpha}_{\lambda} = \gamma^{\mu}_{\lambda}$ and $\gamma^{\mu}_{\lambda} = \delta^{\mu}_{\lambda} + n_{\lambda}n^{\mu}$, we have

$$^{(3)}R_{\sigma\beta} = \gamma^{\nu}_{\beta}\gamma^{\rho}_{\sigma}R_{\rho\nu} + \gamma^{\nu}_{\beta}\gamma^{\rho}_{\sigma}n_{\lambda}n^{\mu}R^{\lambda}_{\rho\mu\nu} + +K^{\alpha}_{\beta}K_{\alpha\sigma} - KK_{\beta\sigma}, \tag{42}$$

which gives us the spatial-spatial projection of the Ricci tensor.