ADM Formulation of Scalar–Tensor–Vector Gravity

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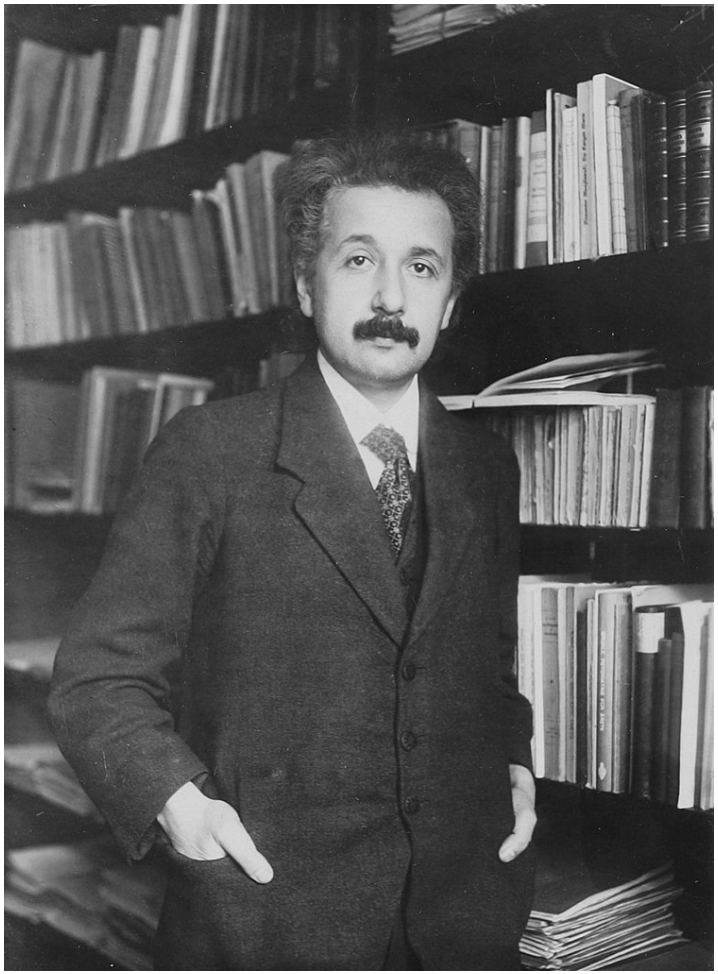
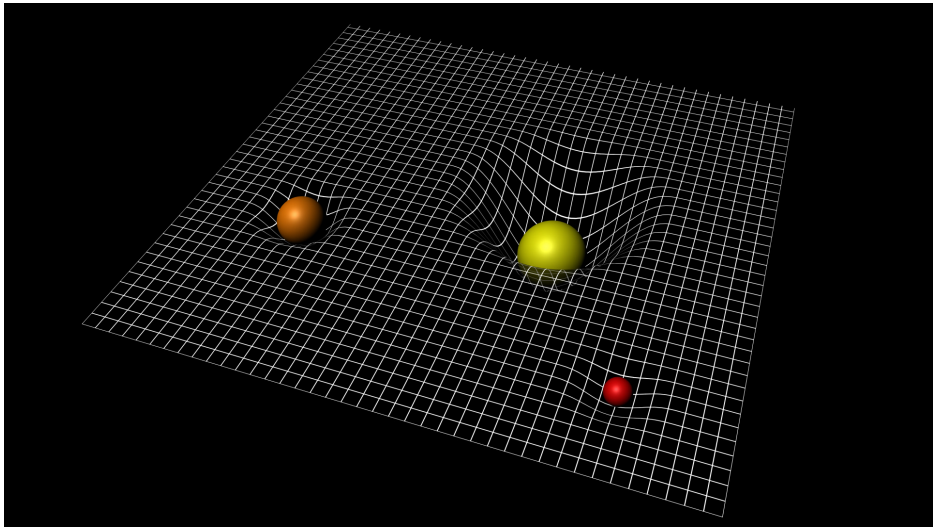
General Relativity





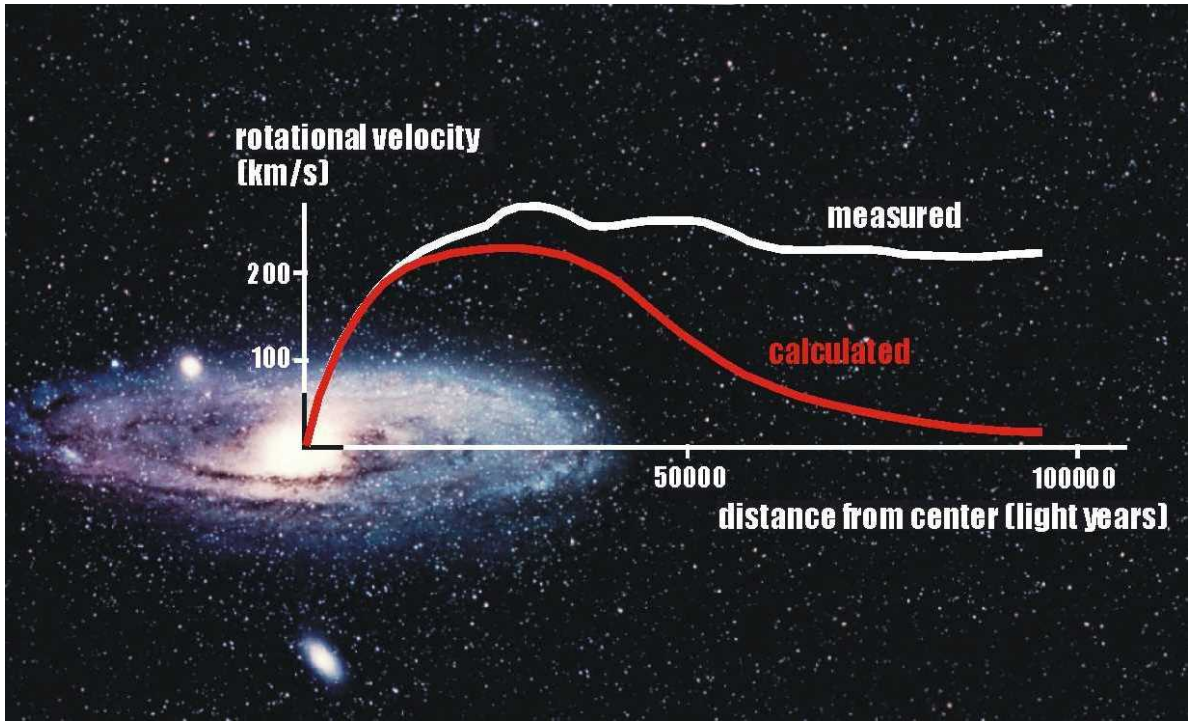
Einstein Field Equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Problems with GR & Modified Gravity

Problems with GR:

- Singularities - Roger Penrose 1965 [1] ∴ quantum gravity?
- Dark matter & dark energy ∴ *modify* general relativity?



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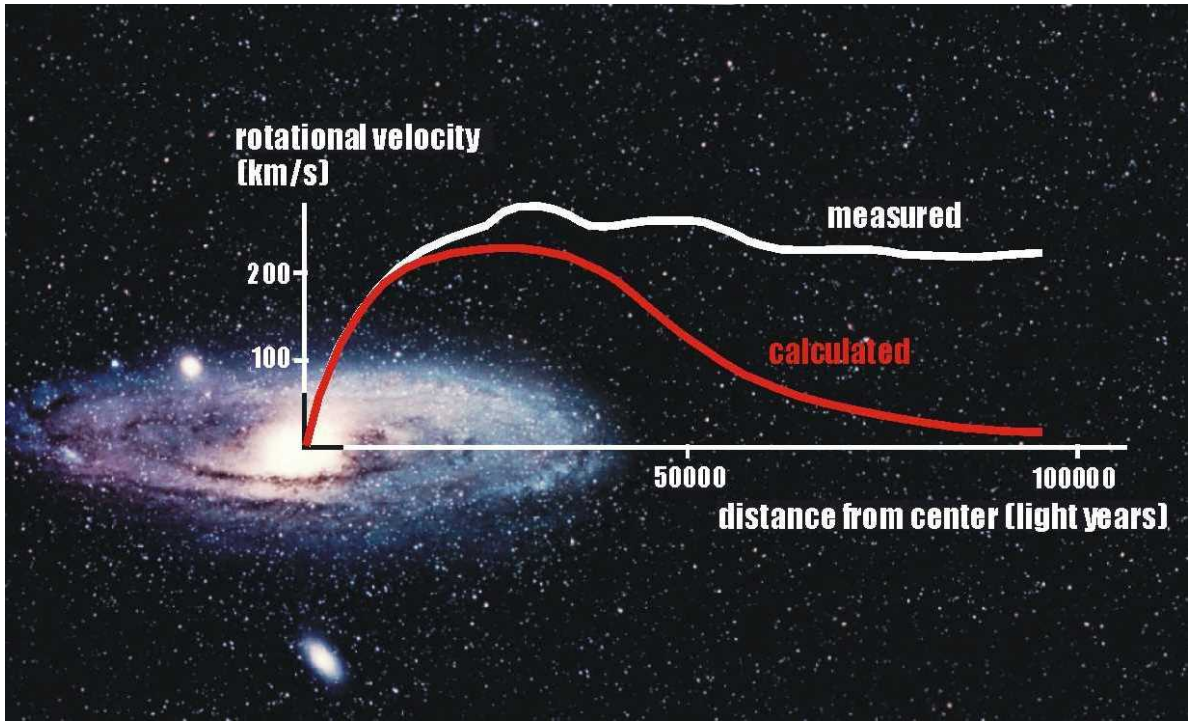

More theories of gravity $\implies$ more field equations to be analyzed

(Provides work for a math thesis)

Lovelock's theorem [2, 3] constrains the space of viable theories.

Scalar–Tensor–Vector Gravity - John Moffat, 2006 [4]:

- Adds vector field ϕ_μ and scalar fields G, μ, ω

$$g_{\mu\nu} \longrightarrow g_{\mu\nu}, \phi_\mu, G, \mu, \omega$$

$$\Longrightarrow \text{ 4 new field equations + modified tensor field equation}$$

G now varies. ϕ_μ adds repulsive gravitational force. μ is mass of ϕ_μ, modifying its strength of interaction. ω changes G.

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STVG tensor field equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} + Q_{\mu\nu} = \frac{8\pi G}{c^4} T'_{\mu\nu}$$

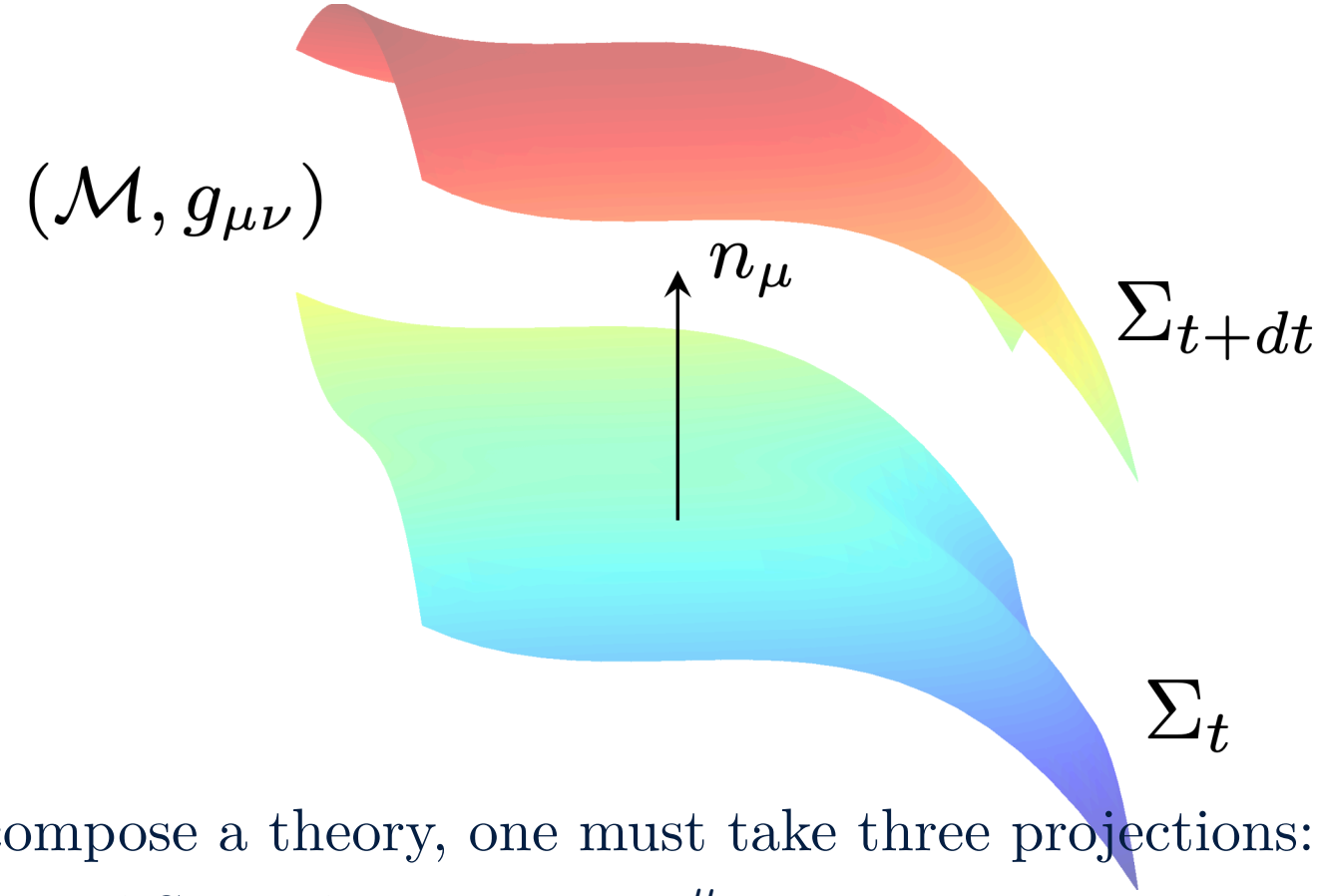
where

$$Q_{\mu\nu} = \left(\nabla^\alpha \nabla_\alpha g_{\mu\nu} \Theta - \nabla_\mu \nabla_\nu \Theta \right) G$$

where $\Theta := 1/G(x)$ and $T'_{\mu\nu} = T^{\text{GR}}_{\mu\nu} + T^{\phi}_{\mu\nu} + T^{G}_{\mu\nu} + T^{\mu}_{\mu\nu} + T^{\omega}_{\mu\nu}$

3+1 Decomposition (Basis of ADM Formalism)

Foliate manifold $\mathcal{M}$ into 3D hypersurfaces Σ_t evolved over time



To decompose a theory, one must take three projections:

- Spatial-Spatial projection; $\gamma_i^\mu \gamma_j^\nu$
- Normal-Normal projection; $n^\mu n^\nu$
- Mixed projection; $\gamma_i^\mu n^\nu$

3+1 Decomposition of STVG

Recall STVG's tensor field equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} + Q_{\mu\nu} = \frac{8\pi G}{c^4} T'_{\mu\nu}$$

where

$$Q_{\mu\nu} = \left(\nabla^\alpha \nabla_\alpha g_{\mu\nu} \Theta - \nabla_\mu \nabla_\nu \Theta \right) G.$$

To compute normal-normal projection:

$$n^\mu n^\nu Q_{\mu\nu} = \left(\nabla^\alpha \nabla_\alpha n^\mu n^\nu g_{\mu\nu} \Theta - \nabla_\mu n^\mu \nabla_\nu n^\nu \Theta \right) G$$

Use $n^\mu n^\nu g_{\mu\nu} = -1$ and $\nabla_\mu n^\mu = \gamma^{\mu\sigma} \nabla_\mu n_\sigma := -K$, so we have

$$n^\mu n^\nu Q_{\mu\nu} = -\left(\nabla^\alpha \nabla_\alpha \Theta + K^2 \Theta \right) G$$

- Computing normal-normal projection of GR field equations results in:

$$ {}^{(3)}R + K^2 - K_{ij}K^{ij} = \frac{16\pi}{c^4}\rho $$

called the **Hamiltonian constraint**; constrains total energy of gravitational field at a given time.

ADM Formulation of STVG

- Computing normal-normal projection of GR field equations results in:

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  called the **Hamiltonian constraint**; constrains total energy of gravitational field at a given time.

- STVG adds a $Q_{\mu\nu}$ term and a modified $T'_{\mu\nu}$, so Hamiltonian constraint for STVG:

$$ {}^{(3)}R + K^2 - K_{ij}K^{ij} - \left(\nabla^\alpha\nabla_\alpha\Theta + K^2\Theta\right)G = \frac{16\pi}{c^4}\left(\rho + \rho'\right) $$

  where I'm suppressing full $\rho' \ldots$ way too long.

- ADM formalism is the collection of theory's projected field equations

Conclusion

- I am applying **3+1 decomposition** to a modified theory of gravity (called **Scalar–Tensor–Vector gravity**)
- The theory is written in an **ADM formulation** [5] when all decompositions are applied
- I see this as a purely mathematical problem:
  I do believe dark matter exists
- Next steps:
  - Finish all three decompositions of the four additional field equations (for ϕ_μ, G, μ, ω fields)
  - Find scenarios where allowed initial conditions of STVG differ from what GR allows?

Sources

[1] Roger Penrose. "Gravitational Collapse and Space-Time Singularities". In: *Phys. Rev. Lett.* (1965).

[2] D. Lovelock. "The Einstein Tensor and its Generalizations". In: *Journal of Mathematical Physics* (1971).

[3] D. Lovelock. "The Four-Dimensionality of Space and the Einstein Tensor". In: *Journal of Mathematical Physics* (1972).

[4] J. W. Moffat. "Scalar–Tensor–Vector Gravity Theory". In: *Journal of Cosmology and Astroparticle Physics* (2006).

[5] R. Arnowitt, S. Deser, and C. W. Misner. "Dynamical Structure and Definition of Energy in General Relativity". In: *Physical Review* (1959).

"Since the mathematicians have invaded the theory of relativity,
I do not understand it myself anymore"
- Albert Einstein