1 Correlation Functions and Power Spectra on 2D Planes

Random variable: a discrete variable whose value is drawn from an underlying distribution. Example: drawing the CMB temperature at random points on the map $T(x_i, y_i)$.

Random field: when the random variable becomes a continuous function across all space considered. Example: having a continuous function that describes CMB temperature $T(x, y)$.

Define the **density contrast** field:

$$\delta(\boldsymbol{x}) := \frac{\delta\rho}{\rho} = \frac{\rho(\boldsymbol{x}) - \langle\rho\rangle}{\langle\rho\rangle} \tag{1}$$

which indicates deviations from the mean matter density in the universe and is defined such that $\langle\delta(x)\rangle = 0$. Defining a quantity whose mean is zero is a useful trick.

Define the **autocorrelation function**:

$$\xi(\boldsymbol{r}) = \langle\delta(\boldsymbol{x})\delta(\boldsymbol{x'})\rangle \tag{2}$$

Note that one can only compute the mean value of a *uncertain* quantity.

The Fourier pair of $\delta(\boldsymbol{x})$ is:

$$\begin{aligned} \delta(\boldsymbol{x}) &= \frac{V}{(2\pi)^3}\int \delta(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}d^3k \\ \delta(\boldsymbol{k}) &= \frac{(2\pi)^3}{V}\int \delta(\boldsymbol{x})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}d^3x \end{aligned} \tag{3}$$

Using these definitions in $\xi(\boldsymbol{r})$:

$$\begin{aligned} \xi(\boldsymbol{r}) &= \left\langle \frac{V^2}{(2\pi)^6}\int \delta(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}d^3k \int \delta(\boldsymbol{k'})e^{i\boldsymbol{k'}\cdot\boldsymbol{x'}}d^3k' \right\rangle \\ &= \frac{V^2}{(2\pi)^6}\int e^{i\boldsymbol{k}\cdot\boldsymbol{x}}d^3k \int e^{i\boldsymbol{k'}\cdot\boldsymbol{x'}} \left\langle\delta(\boldsymbol{k})\delta(\boldsymbol{k'})\right\rangle d^3k' \\ &= \frac{V^2}{(2\pi)^6}\int e^{i\boldsymbol{k}\cdot\boldsymbol{x}}d^3k \int e^{i\boldsymbol{k'}\cdot\boldsymbol{x'}} \Big((2\pi)^3\delta^D(\boldsymbol{k}+\boldsymbol{k'})P(k)\Big)d^3k' \\ &= \frac{V}{(2\pi)^3}\int e^{i\boldsymbol{k}\cdot\boldsymbol{x}}e^{-i\boldsymbol{k}\cdot\boldsymbol{x'}}P(k)d^3k \\ &= \frac{V}{(2\pi)^3}\int e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}P(k)d^3k \end{aligned} \tag{4}$$

Which tells us that

$$P(k) = \frac{(2\pi)^3}{V} \int e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \xi(\boldsymbol{r}) d^3r \tag{5}$$

Since we assume isotropy of the CMB, we know that $\boldsymbol{k}$ will only depend on its magnitude k and not its direction. This tells us to switch to spherical coordinates (we're in 3D): $\langle k_x, k_y, k_z \rangle \to \langle k, \theta, \phi \rangle$ where $\theta : [0, \pi]$ and $\phi : [0, 2\pi]$ as usual. Recall the transformation $dk_x dk_y dk_z = k^2 \sin\theta dk d\theta d\phi$ and $\boldsymbol{r} \cdot \boldsymbol{k} = kr\cos\theta$:

$$\begin{aligned}
\xi(\boldsymbol{r}) &= \frac{V}{(2\pi)^3} \int_0^\infty \int_0^\pi \int_0^{2\pi} e^{-ikr\cos\theta} P(k) k^2 \sin\theta d\phi d\theta dk \\
&= \frac{V}{(2\pi)^3} \int_0^\infty \int_0^\pi 2\pi k^2 P(k) e^{-ikr\cos\theta} \sin\theta d\theta dk \\
&= \frac{V}{(2\pi)^3} \int_0^\infty 2\pi k^2 P(k) \left(\frac{e^{ikr} - e^{-ikr}}{ikr} \right) dk \\
&= \frac{V}{(2\pi)^3} \int_0^\infty 2\pi k^2 P(k) \frac{2\sin(kr)}{kr} dk \\
&= \frac{V}{(2\pi)^3} \int_0^\infty P(k) \frac{\sin(kr)}{kr} 4\pi k^2 dk
\end{aligned} \tag{6}$$

Effort has gone into predicting the realized form of $P(k)$ in cosmology, but such efforts are limited by our lack of observational data on the early universe. As such, $P(k)$ is typically assumed to be a featureless power law:

$$P(k) \propto k^n \tag{7}$$

For reasons I have to think about more, a dimensionless power spectrum is defined:

$$\Delta^2(k) = \frac{V}{(2\pi)^3} 4\pi k^3 P(k) \tag{8}$$

$$\xi(\boldsymbol{r}) = \int_0^\infty \Delta^2(k) \frac{\sin(kr)}{r} dk \tag{9}$$

1.1 Simulating a Gaussian Random Field on a 2D Plane

Because we wish to simulate the Gaussian random field, our goal is to simulate $\delta(x, y)$.

$$P(k) = \frac{1}{k_x^2 + k_y^2} \tag{10}$$

$$\delta(x, y) = \sum_{i,j} A_{ij} e^{i(k_x x + k_y y)} \tag{11}$$

where

$$A_{ij} = \sqrt{\frac{P(k_i, k_j)}{2}} (a_{ij} + ib_{ij}) \tag{12}$$

2 Correlation Functions and Power Spectra on 2D Spheres

The CMB does not actually exist on a 2D plane, though, so we need a way of writing correlation functions and power spectra on a 2D sphere. We are specifically interested in the CMB temperature as a function of sky location $T(\boldsymbol{n})$, where $\boldsymbol{n}$ is a vector drawn from the center of a CMB sky map. The corelation function will be defined as

$$C(\theta) = \langle T(\boldsymbol{n}) T^*(\boldsymbol{n'}) \rangle \tag{13}$$

where $\theta = \arccos(\boldsymbol{n} \cdot \boldsymbol{n'})$.

Just as in the planar case, we express our functions $T(\boldsymbol{n})$ as an expansion of basis functions. Since we are on a sphere and the basis functions of a sphere are spherical harmonics, we write

$$T(\boldsymbol{n}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\boldsymbol{n}) \tag{14}$$

where $Y_{\ell m}(\boldsymbol{n})$, the spherical harmonics, are the spherical analogs of $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ basis functions.

To motivate the form of $a_{\ell m}$, recall that the planar coefficients in the Fourier basis $f(x) = \sum_n A_n \exp(inx)$ are

$$A_n = \int f(x) e^{-inx} dx. \tag{15}$$

Similarly, the $a_{\ell m}$ spherical coefficients are defined as

$$a_{\ell m} = \int T(\boldsymbol{n}) Y^*_{\ell m}(\boldsymbol{n}) d\Omega_{\boldsymbol{n}}. \tag{16}$$

Using the definition of $T(\boldsymbol{n})$ written in the spherical harmonic basis, we can now simplify our correlation function expression:

$$\begin{aligned} C(\theta) &= \left\langle \sum_{\ell m} a_{\ell m} Y_{\ell m}(\boldsymbol{n}) \sum_{\ell' m'} a_{\ell' m'} Y^*_{\ell' m'}(\boldsymbol{n}) \right\rangle \\ &= \sum_{\ell m} \sum_{\ell' m'} Y_{\ell m} Y^*_{\ell' m'} \langle a_{\ell m} a_{\ell' m'} \rangle \\ &= \sum_{\ell m} \sum_{\ell' m'} Y_{\ell m} Y^*_{\ell' m'} \Big(\delta_{\ell\ell'} \delta_{mm'} C(\ell) \Big) \\ &= \sum_{\ell m} Y_{\ell m} Y^*_{\ell m} C(\ell) \end{aligned} \tag{17}$$

We finish by using the **Addition Theorem for Spherical Harmonics**:

$$P_\ell(\cos\theta) = \frac{4\pi}{2\ell+1} \sum_m Y_{\ell m}(\boldsymbol{n}) Y^*_{\ell m}(\boldsymbol{n'}) \tag{18}$$

where P_ℓ is the Legendre polynomial of degree ℓ. Using this fact gives us

$$C(\theta) = \frac{1}{4\pi}\sum_{\ell}(2\ell+1)P_\ell(\cos\theta)C(\ell) \tag{19}$$

2.1 Simulating a Gaussian Random Field on a 2D Sphere

$$T(\boldsymbol{n}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}Y_{\ell m}(\boldsymbol{n}) \tag{20}$$

where

$$a_{\ell m} = \sqrt{\frac{C(\ell)}{2}}\Big(a_{ij} + ib_{ij}\Big) \tag{21}$$

where a_{ij} and b_{ij} are random variables pulled from a Gaussian distribution. And:

$$Y_{\ell m} = ? \tag{22}$$

3 Relating Angular Power Spectrum $C(\ell)$ to Power Spectrum $P(k)$ + Some Fourier Math

3.1 Fourier Transform of $f'(x)$

Deriving an expression for $\mathscr{F}\{f'(x)\}$ will be useful for the proceeding content:

$$\begin{aligned}\mathscr{F}\{f'(x)\} &= \int_{-\infty}^{\infty} e^{ikx}\frac{d}{dx}f'(x)dx \\ &= e^{ikx}f(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)ike^{ikx}dx \\ &= -ik\int_{-\infty}^{\infty} e^{ikx}f(x)dx = -ikf(k)\end{aligned} \tag{23}$$

3.2 test

It's given in Peebles that (don't yet know how to get here):

$$\tau := \frac{\delta T}{T} = -\frac{1}{3}\frac{G\rho_0}{D_0}\int \frac{\delta(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3r' \tag{24}$$

where ρ_0 is ..., D_0 is ... , and $\delta(\boldsymbol{r}')$ is the mass density contrast at some point $\boldsymbol{r}'$. If we then express $\delta(\boldsymbol{r}')$ as a Fourier integral, we have

$$\tau = -\frac{1}{3}\frac{G\rho_0}{D_0}\int \delta(\boldsymbol{k})\bigg(\int \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}'}}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3r'\bigg)d^3k \tag{25}$$

To evaluate this, consider $f(\boldsymbol{r}') = \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}$ and take:

$$\begin{aligned} \mathscr{F}\{\nabla^2 f(\boldsymbol{r}')\} &= \int e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \nabla^2 f(\boldsymbol{r}') d^3r' = -4\pi \int e^{i\boldsymbol{k}\cdot\boldsymbol{r}'} \delta(\boldsymbol{r}-\boldsymbol{r}') d^3r' \\ &= -k^2 \mathscr{F}\{f(\boldsymbol{r}')\} = -4\pi e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \\ &\implies \mathscr{F}\{f(\boldsymbol{r}')\} = \frac{4\pi e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{k^2} \end{aligned} \tag{26}$$

This makes out τ expression:

$$\tau = -\frac{4\pi}{3}\frac{G\rho_0}{D_0}\int \delta(\boldsymbol{k})\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{k^2} d^3k \tag{27}$$

We now define $\Omega H_0^2 := \frac{8}{3}G\rho_0$ for reasons I don't understand. This makes our expression for temperatures fluctuations:

$$\tau = -\frac{\Omega H_0^2}{2D_0}\int \delta(\boldsymbol{k})\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{k^2} d^3k \tag{28}$$

To begin relating $C(\ell)$ to $P(k)$, we will first use the plane wave expression for $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$:

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} i^\ell j_\ell(\boldsymbol{k}\cdot\boldsymbol{r}) Y_{\ell m}(\hat{\boldsymbol{r}}) Y^*_{\ell m}(\hat{\boldsymbol{k}}) \tag{29}$$

This makes τ:

$$\tau = -\frac{4\pi\Omega H_0^2}{2D_0}\int \frac{\delta(\boldsymbol{k})}{k^2} \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} i^\ell j_\ell(\boldsymbol{k}\cdot\boldsymbol{r}) Y_{\ell m}(\hat{\boldsymbol{r}}) Y^*_{\ell m}(\hat{\boldsymbol{k}}) d^3k \tag{30}$$

We will now use

$$a_{\ell m} = \int \tau Y^*_{\ell m} d\Omega \tag{31}$$

since $\langle |a_{\ell m}|^2 \rangle$ will give us the angular power spectrum $C(\ell)$, and we have a $\delta(\boldsymbol{k})$ inside of the expression for τ that will eventually give us a $P(k)$ term when we take $\langle |a_{\ell m}|^2 \rangle$. Using this fact (I actually don't know what the introduced $Y^*_{\ell m}$ is a function of... I'll guess $\boldsymbol{n}$ but I don't know how this then matches the existing $Y_{\ell m}(\boldsymbol{r})$):

$$\begin{aligned} a_{\ell m} &= -\frac{2\pi\Omega H_0^2}{D_0}\int \frac{\delta(\boldsymbol{k})}{k^2} \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} i^\ell j_\ell(\boldsymbol{k}\cdot\boldsymbol{r}) Y^*_{\ell m}(\hat{\boldsymbol{k}}) \left(\int Y_{\ell m}(\boldsymbol{r}) Y^*_{\ell m}(\boldsymbol{n}) d\Omega\right) d^3k \\ &= -\frac{2\pi\Omega H_0^2}{D_0}\int \frac{\delta(\boldsymbol{k})}{k^2} \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} i^\ell j_\ell(\boldsymbol{k}\cdot\boldsymbol{r}) Y^*_{\ell m}(\hat{\boldsymbol{k}}) \delta_{\ell\ell'}\delta_{mm'} d^3k \\ &= -\frac{2\pi\Omega H_0^2}{D_0}\int \frac{\delta(\boldsymbol{k})}{k^2} \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} i^\ell j_\ell(\boldsymbol{k}\cdot\boldsymbol{r}) Y^*_{\ell m}(\hat{\boldsymbol{k}}) d^3k \end{aligned} \tag{32}$$

Now we take $\langle |a_{\ell m}|^2 \rangle$ to arrive at $C(\ell)$:

$$\begin{aligned}
C(\ell) = \langle |a_{\ell m}|^2 \rangle &= \frac{4\pi^2\Omega^2 H_0^4}{D_0^2} \int\int \frac{1}{k^2}\frac{1}{k'^2} \sum_{\ell m}\sum_{\ell' m'} i^\ell j_\ell(\boldsymbol{k}\cdot\boldsymbol{r}) i^{\ell'} j_{\ell'}(\boldsymbol{k'}\cdot\boldsymbol{r'}) Y^*_{\ell m} Y_{\ell' m'} \langle \delta(\boldsymbol{k})\delta(\boldsymbol{k'}) \rangle d^3k d^3k' \\
&= \frac{4\pi^2\Omega^2 H_0^4}{D_0^2} \int\int \frac{1}{k^2}\frac{1}{k'^2} \sum_{\ell m}\sum_{\ell' m'} i^\ell j_\ell(kr) i^{\ell'} j_{\ell'}(k'r) Y^*_{\ell m} Y_{\ell' m'} \delta^3(\boldsymbol{k}-\boldsymbol{k'}) P(k) d^3k' d^3k \\
&= \frac{4\pi^2\Omega^2 H_0^4}{D_0^2} \int \frac{1}{k^2} \sum_{\ell m}\sum_{\ell' m'} i^\ell j_\ell(kr) i^{\ell'} j_{\ell'}(kr) Y^*_{\ell m} Y_{\ell' m'} P(k) d\Omega dk \\
&= \frac{4\pi^2\Omega^2 H_0^4}{D_0^2} \int \frac{1}{k^2} P(k) \sum_{\ell m}\sum_{\ell' m'} i^\ell j_\ell(kr) i^{\ell'} j_{\ell'}(kr) \left(\int Y^*_{\ell m} Y_{\ell' m'} d\Omega \right) dk \\
&= \frac{4\pi^2\Omega^2 H_0^4}{D_0^2} \int \frac{1}{k^2} P(k) \sum_{\ell m}\sum_{\ell' m'} i^\ell j_\ell(kr) i^{\ell'} j_{\ell'}(kr) \delta_{\ell\ell'}\delta_{mm'} dk \\
&= \frac{4\pi^2\Omega^2 H_0^4}{D_0^2} \int \frac{1}{k^2} P(k) \sum_{\ell} i^\ell j_\ell(kr) i^{\ell} j_{\ell}(kr) dk \\
&= \frac{4\pi^2\Omega^2 H_0^4}{D_0^2} \int \frac{1}{k^2} P(k) j_\ell^2(kr) dk
\end{aligned} \tag{33}$$

So we have

$$\boxed{C(\ell) = \frac{4\pi^2\Omega^2 H_0^4}{D_0^2} \int \frac{1}{k^2} P(k) j_\ell(kr)^2 dk} \tag{34}$$

... given a power spectrum $P(k)$, we can generate an angular power spectrum $C(\ell)$ using this relation!

In general, a **transfer function** $W_\ell(k)$ is defined, which describes how fluctuations at different scales k are mapped to angular scales on the sky ℓ. So we have a transfer function:

$$W_\ell(k) = \frac{4\pi^2\Omega^2 H_0^2}{D_0^2} \frac{j_\ell(kr)^2}{k^2} \tag{35}$$

(maybe look into other transfer functions now and understand when the one above applies?)