1 Closed Quantum Mechanics

1.1 Hilbert space, wavefunctions, and operators

Up until the early twentieth century, all observable quantities about a system were thought to be single-valued and continuous; that if observed, a system would only exist in a single value of that variable and if it were to change, it would do so continuously. All of the physics prior to the twentieth century was therefore carried out in functional representations; functions gave physicists exact values for a system's quantities and continuous trajectories of them to describe nature by. Beginning with Max Planck's heuristic derivation of the law of blackbody radiation and Einstein's discovery of the photoelectric effect though, it began to be realized that the observed values of these variables for a quantum system were discrete and were probabilistically determined. In order to construct physics for quantum systems, it was realized that linear algebra would be a necessary framework for doing so.

Because the values for physical quantities were found to be discrete and whose values for those quantities were probabilistically determined upon measurement, it was realized that the state of a quantum particle was required to be described by a linear superposition of all possible states with constants that described the probabilistic amplitude of each state. If a quantum particle has a set of energies $\{|E_i\rangle\}$, each with probability amplitude c_i, the actual probability of observing this particle with energy state $|E_i\rangle$ is the squared modulus of its probabilistic weight, $c_i\bar{c}_i$. For example, if we were only interested in a single particle's energy levels, the full state of the particle, $|\psi\rangle$, would be expressed as:

$$|\psi\rangle = c_1 |E_1\rangle + c_2 |E_2\rangle + \cdots + c_n |E_n\rangle = \sum_i c_i |E_i\rangle \tag{1}$$

In the language of linear algebra, the state $|\psi\rangle$ of a quantum particle is represented by a vector in some vector space where the linearly independent components of these vectors, $E_1, E_2, \ldots, E_n$ are the basis vector states for that quantity; they represent the definite, possible values that the quantity E could take on that are mutually independent. The number n of linearly independent vectors needed to describe the state of a quantum system form the basis for the Hilbert space $\mathcal{H}$ of the system, a generalized vector space on which the vectors describing the state of our system, $|\psi\rangle$, act.

In quantum mechanics, the language used to describe quantities such as position and momentum indicates them as **observables** since these are the quantities of the quantum system that could be physically observed. The most obvious observables of a quantum system are its energy E, angular momentum L, position x, and momentum p. All of these observables can be expressed as a linear combination of their basis vector states.

An **operator** $\hat{\mathcal{O}}$ is a function that linearly maps vectors to vectors in a specified Hilbert space. Because it is a linear map, its action on the basis states $\{|\psi_i\rangle\} \in \mathcal{H}$ of a quantum system is all that is necessary to describe its action on the full state $|\psi\rangle$:

$$\hat{\mathcal{O}} |\psi\rangle = \hat{\mathcal{O}} \left(\sum_i c |\psi_i\rangle \right) = \sum_i c_i \hat{\mathcal{O}} |\psi_i\rangle . \tag{2}$$

1.2 Unitaries and time evolution

Maybe the most important class of operators are those that are **unitary**, which satisfy the condition

$$U^\dagger U = U U^\dagger = \mathbb{1}. \tag{3}$$

These unitary operators are usually used to describe the time evolution of a quantum system:

$$|\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle \tag{4}$$

Note that this definition implicitly requires that $U^\dagger = U$, which gives U the property of being Hermitian. Unitary operators are extremely important to quantum mechanics because they preserve one's ability to talk about a system's state probabilities. Without unitarity, one could apply an operator to a quantum system and not get a valid state in return, violating all interpretation of the wavefunction. To see how a unitary operator preserved probability, recall that the inner product of any vector with itself will always be equal to one; $\langle\psi|\psi\rangle = 1$. If we let $\langle\psi'| = \langle\psi| U^\dagger$ and $|\psi'\rangle = U |\psi\rangle$, then we have

$$\langle\psi'|\psi'\rangle = \langle\psi| UU^\dagger |\psi\rangle = \langle\psi| \mathbb{1} |\psi\rangle = \langle\psi|\psi\rangle = 1. \tag{5}$$

We can also show that the eigenvalues of a unitary operator must always have a magnitude of one. We do this with the eigendecomposition of

$$U |\psi\rangle = \lambda |\psi\rangle \tag{6}$$

and

$$\langle\psi| U^\dagger = \langle\psi| \lambda^\dagger. \tag{7}$$

Multiplying these expressions together, we have

$$\langle\psi| U^\dagger U |\psi\rangle = \langle\psi| \lambda^\dagger \lambda |\psi\rangle \tag{8}$$

and since $U^\dagger U = 1$, this requires that $\lambda^\dagger\lambda \equiv |\lambda|^2 = 1 \implies |\lambda| = 1$.

Because of the fact that $|\lambda| = 1$ for all eigenvalues of a unitary matrix, it is convenient to write the operator as

$$U = \sum_k e^{i\lambda_k} |k\rangle\langle k| \tag{9}$$

where $|k\rangle$ is one of the eigenstates **(OF WHAT?)**. Doing this allows us to easily write

$$U(t) |k\rangle = e^{i\lambda_k(t)} |k\rangle . \tag{10}$$

(I don't understand this... how can an eigenvalue change with time? ... I must not understand what should be in the exponent there. The book has it written as exp($i\alpha_k$)).

If we have a quantum system evolving in time by some unitary, $|\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle$, we may want to know how the wavefunction itself is changing. Because the derivative of a vector is also a vector, there must exist an operator $\hat{G}$ such that

$$\hat{G} |\psi\rangle = \frac{d}{dt} |\psi\rangle \tag{11}$$

It's common to consider the Hamiltonian operator **(why is this? The book doesn't seem to give much motivation for this)** $\hat{H} = i\hbar\hat{G}$ so that

$$\hat{H} |\psi\rangle = i\hbar \frac{d}{dt} |\psi\rangle . \tag{12}$$

This is called the Schrödinger equation. Written in this way, it is just the definition of the operator $\hat{H}$. We can apply the Schrödinger equation to a unitary $U(t)$ to find that

$$\hat{H}U(t, t_0) |\psi(t_0)\rangle = i\hbar \frac{d}{dt} U(t, t_0) |\psi(t_0)\rangle . \tag{13}$$

We can then remove $|\psi(t_0)\rangle$ from this expression, resulting in

$$\hat{H}U(t) = i\hbar \frac{d}{dt} U(t) \tag{14}$$

which has the solution

$$\boxed{U(t) = e^{-i\hat{H}t/\hbar}} \tag{15}$$

We can therefore use the Hamiltonian operator of the system to describe its time evolution.

1.3 Composite systems and interaction

When describing a quantum system, it may have distinct components that feature their own, independent Hilbert spaces. To describe the full composite system from the Hilbert spaces of the subsystems, the tensor product is used,

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B. \tag{16}$$

Not all composite systems can be considered as a tensor product between its components. Those that can be are called product states. Consider the example of the Bell state, defined as

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) = \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} . \tag{17}$$

Because there is no possible way of generating the last matrix from a tensor product of $|0\rangle$ and $|1\rangle$ states, it isn't a product state.

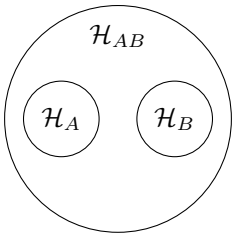

Figure 1: A scenario of two independent subsystems in a composite system. Each subsystem would has its own Hilbert space in isolation so we must describe the full system by $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

If you had a composite system of two particles, one of which was in the state $|u_1\rangle$ while the other was in state $|v_1\rangle$, the state of your composite system would just be $|u_1\rangle \otimes |v_1\rangle$. Similarly, if one particle was in state $\alpha_1 |u_1\rangle + \alpha_2 |u_2\rangle$ and the other was in state $\beta_1 |v_1\rangle + \beta_2 |v_2\rangle$, your composite system would be in the state

$$(\alpha_1 |u_1\rangle + \alpha_2 |u_2\rangle) \otimes (\beta_1 |v_1\rangle + \beta_2 |v_2\rangle) \tag{18}$$

$$= \alpha_1\beta_1 |u_1\rangle \otimes |v_1\rangle + \alpha_1\beta_2 |u_1\rangle \otimes |v_2\rangle + \alpha_2\beta_1 |u_2\rangle \otimes |v_1\rangle + \alpha_2\beta_2 |u_2\rangle \otimes |v_2\rangle$$

That is, if we were told that the state of a composite system of two qubits was either in $|u_1\rangle \otimes |v_1\rangle$ or $\alpha_1\beta_1 |u_1\rangle \otimes |v_1\rangle + \alpha_1\beta_2 |u_1\rangle \otimes |v_2\rangle + \alpha_2\beta_1 |u_2\rangle \otimes |v_1\rangle + \alpha_2\beta_2 |u_2\rangle \otimes |v_2\rangle$, we would be able to work backwards to determine the exact state of each individual particle in the subsystem. These two examples of composite systems exist in product states since, given the full state $\mathcal{H}_{AB}$, we are able to rewrite this composite system as a product of its components' Hilbert spaces, $\mathcal{H}_A \otimes \mathcal{H}_B$ to individually tell us about the Hilbert spaces of each particle, $\mathcal{H}_A$ and $\mathcal{H}_B$.

When a composite state can't be expressed as a product state, it is called an **entangled state**. If a composite system is in an entangled state, it isn't always possible to assign state vectors to the individual subsystems. Consider the scenario if we were told that a composite of two subsystems, where the subsystem has basis states $|u_1\rangle, |u_2\rangle$ and the second has $|v_1\rangle, |v_2\rangle$, was described by the state

$$|u_1\rangle \otimes |v_1\rangle + |u_2\rangle \otimes |v_2\rangle . \tag{19}$$

There is no way of representing the state of the individual subsystems given this composite system state; we can't write anything of the form

$$|u_1\rangle \otimes |v_1\rangle + |u_2\rangle \otimes |v_2\rangle = (\dots)_{|u\rangle} \otimes (\dots)_{|v\rangle} \tag{20}$$

to inform us of what the individual components' quantum states look like. This is what we mean by an entangled state. We can't separate states of an entangled composite system to just the components of the individual subsystem states. We cannot separate entangled states in this way because they are sharing states with each other.

This is where quantum mechanics clearly diverges from classical mechanics; to describe a composite system in classical mechanics, you would need to specify the state of each component, such as their position and momentum. We are now saying that to describe a quantum system, you don't need information about the exact state of the subsystems; in fact, you *can't* always have it. To describe a system made up of multiple particles, we don't need to and often can't specify the state of each particle, as is necessary in classical mechanics, in order to describe a composite system.

We define a system to be non-interacting if the composite Hamiltonian is just the sum of the subsystem Hamiltonians:

$$H_{AB} = H_A + H_B. \tag{21}$$

This tells us that subsystems are not interacting if the sum of their total energies equals the total energy of the composite system. If the subsystems are interacting, they may share some energy, meaning that $H_{AB} < H_A + H_B$. **(it's not ever the case that $H_{AB} > H_A + H_B$, right?)**

1.4 Density operators and their time evolution

When subsystems are entangled, it is often not possible to write down a complete wavefunction $|\psi\rangle$ for each of them. Entangled subsystems are capable of sharing and exchanging states with each other so it's not possible to describe the state of a subsystem as being identical to the subsystem's wavefunction in isolation. Entanglement requires that each subsystem carries information about other subsystems.

To describe the state of an entangled quantum system then, we need to first describe the full system as a mixture of states provided by the subsystems. The set of states describing a composite system don't need to be orthogonal to or independent of each other since its subsystems may share states in isolation. To account for a composite system's mixed states $|\psi_i\rangle\langle\psi_i|$ and their probabilities p_i, the density matrix

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{22}$$

is defined to describe the state of a composite quantum system.

For an example that will be useful later, $\{\mathbb{1}, \sigma_X, \sigma_Y, \sigma_Z\}$ forms a basis for a qubit system (which has a two-dimensional Hilbert space). Due to the requirement that a valid density operator satisfies $\mathrm{Tr}[\rho] = 1$, and because $\mathrm{Tr}[\mathbb{1}] = 1$ and $\mathrm{Tr}[\sigma_i] = 0$, we can write any density matrix for a qubit system in the form

$$\rho = \frac{1}{2}(1 + \sum_i a_i \sigma_i). \tag{23}$$

Given ρ, you can determine the a_i values by using

$$a_i = \mathrm{Tr}[\rho\sigma_i]. \tag{24}$$

Because we stated a wavefunction evolves by a unitary operator

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle\,, \tag{25}$$

pure state density matrices, $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$, will evolve by

$$\rho(t) = U(t)\rho(0)U^\dagger(t). \tag{26}$$

To find the differential equation governing the a pure state's motion, we take

$$\frac{d}{dt}\rho = \frac{d}{dt}(|\psi\rangle\langle\psi|) = \frac{d\,|\psi\rangle}{dt}\,\langle\psi| + |\psi\rangle\,\frac{d\,\langle\psi|}{dt} \tag{27}$$

and since we said that

$$\hat{H}\,|\psi\rangle = i\hbar\frac{d}{dt}\,|\psi\rangle \tag{28}$$

from the Schrödinger equation, we have that

$$\begin{aligned} \frac{d\rho}{dt} &= \frac{i}{\hbar}\hat{H}\,|\psi\rangle\,\langle\psi| - \frac{i}{\hbar}|\psi\rangle\langle\psi|\hat{H} \equiv -\frac{i}{\hbar}[\hat{H}, \rho] \\ &\implies i\hbar\frac{d\rho}{dt} = [\hat{H}, \rho], \end{aligned} \tag{29}$$

which is called the Von Neumann equation of motion.

2 Open QM

2.1 Open system dyanamics

The previous material considered quantum mechanics in a *closed* system; one which has no interactions with its environment. In such a situation, we can either define a full wavefunction or density matrix for the state of the system and evolve it in time with the tools described previously. Though the assumption that a system will not interact with its environment can occasionally be safe to make, we know it slightly misrepresents the underlying physical phenomenon at best and is completely inadequate to describe scenarios where a quantum system has a high degree of interaction with its environment at worst. We know that most realistic quantum mechanical systems do interact with their environment so it's crucial to develop the theory of systems that are allowed to have this interaction. These systems are called *open* and we have already covered the primary tool for describing the state of a system embedded in an environment it is interacting with, the density matrix.

To describe open quantum systems, we have to assume that the environment it's embedded in is itself a closed system, which is again a limiting assumption but is not nearly as inaccurate as assuming that the system itself is closed. If there is no *initial* interaction between the system and environment, we know we can represent the full

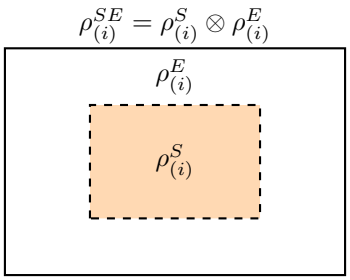

Figure 2: An open system embedded in a closed environment. There is assumed to be no initial entanglement or interaction between the system and environment, so we can write the initial state as a tensor product $\rho^{SE}_{(i)} = \rho^{S}_{(i)} \otimes \rho^{E}_{(i)}$.

system + environment state as a tensor product: Exactly as we began describing the time evolution of density matrices in the previous section, we can describe the time evolution of the full system + environment with some unitary acting on the initially unentangled state:

$$\rho^{SE}(t) = U(t)\left(\rho^{S}_{(i)} \otimes \rho^{E}_{(i)}\right) U^{\dagger}(t). \tag{30}$$

Given this, we may only wish to describe the time evolution of the system instead of describing how the state of the system *and* environment evolve. To do this, we take the partial trace over the environment of the previous expression, which sums over all possible states of the environment, effectively averaging their effect on the system out. Let $\{\psi_i\}$ be the set of basis states for the environment and let the initially pure state of the environment be $\rho^{E}_{(i)} = \sum_j |\psi_j\rangle\langle\psi_j|$. Then computing the partial trace results in

$$\begin{aligned}
\rho^{S}(t) &= \mathrm{Tr}_E\left(U(t)\left(\rho^{SE}_{(i)}\right)U^{\dagger}(t)\right) \\
&= \mathrm{Tr}_E\left(U(t)\left(\rho^{S}_{(i)} \otimes \rho^{E}_{(i)}\right)U^{\dagger}(t)\right) \\
&= \sum_i \langle\psi_i| U(t)\left(\rho^{S}_{(i)} \otimes \rho^{E}_{(i)}\right) U^{\dagger}(t) |\psi_i\rangle \\
&= \sum_i \langle\psi_i| U(t)\left(\rho^{S}_{(i)} \otimes \sum_j |\psi_j\rangle\langle\psi_j|\right) U^{\dagger}(t) |\psi_i\rangle \\
&= \sum_{i,j} \langle\psi_i| U |\psi_j\rangle \rho^{S}_{(i)} \langle\psi_j| U^{\dagger} |\psi_i\rangle .
\end{aligned} \tag{31}$$

What's typically done here is to define $K_\alpha := \langle\psi_i| U |\psi_j\rangle$, called Kraus operators, so we can express this more compactly as

$$\boxed{\rho^{S}(t) = \sum_\alpha K_\alpha \rho^{S}_{(i)} K^{\dagger}_\alpha} \tag{32}$$

where, again, $\{|\psi_i\rangle\}$ is the set of basis states of the environment which are being averaged over and $\{|\psi_j\rangle\}$ is the initial state of the environment.

It's important to recognize that Kraus operators are not necessarily unitary ($K^{\dagger}_\alpha K_\alpha = \mathbb{1}$ isn't always true for a specific α) but

$$\sum_\alpha K^{\dagger}_\alpha K_\alpha = \mathbb{1} \tag{33}$$

must be true. Consider the amplitude damping channel, which models energy dissipation of a system such as a photon being spontaneously released from the system. The Kraus operators for this single qubit channel are defined as

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} \tag{34}$$

and one can show by computation that $K^{\dagger}_0 K_0, K^{\dagger}_1 K_1 \neq \mathbb{1}$ but $K^{\dagger}_0 K_0 + K^{\dagger}_1 K_1 = \mathbb{1}$.

This is important to mention because the non-unitarity of Kraus operators mean that we are now capable of describing processes which are non-unitary. Non-unitary processes describe situations where initially unentangled states become entangled and your system can now freely interact with its environment.

A mapping that takes an intial state of a quantum system to a later state in time, regardless of whether or not that mapping describes unitary evolution or not, is called a *quantum channel*, typically denoted as $\Lambda[\rho]$. So we have that

$$\Lambda_S[\rho^{SE}_{(i)}] = \sum_\alpha K_\alpha \rho^S_{(i)} K^\dagger_\alpha = \rho^S(t) \tag{35}$$

where $\Lambda_S[\cdot]$ is used to indicate that we trace out the environment in this mapping. This is demonstrated in the following figure:

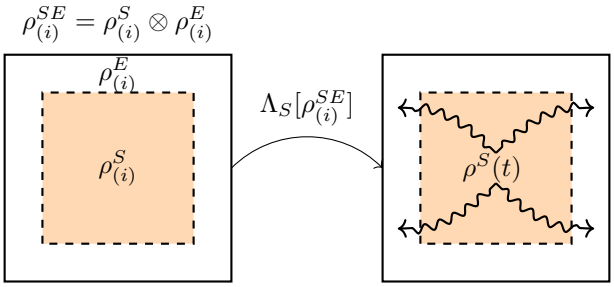

Figure 3: An initially unentangled system and environment $\rho^{SE}_{(i)}$ is allowed interact and become entangled after a quantum channel $\Lambda_S[\rho^{SE}_{(i)}]$ is applied to extract the dynamics of the system $\rho^S(t)$.

2.2 NCP maps

In arriving at the Kraus representation for the map $\Lambda_S[\rho^{SE}] = \rho^S(t)$ which was capable of averaging over the influence of the environment so that we could describe the dynamics of just the system, we had to assume that $\rho^{SE}_{(i)} = \rho^S_{(i)} \otimes \rho^E_{(i)}$, meaning that the system and environment were assumed to be initially unentangled. For this scenario, the resulting density matrix $\rho^S(t)$ will always be a valid quantum mechanical state. We must revisit the assumption that the initial state of the system and environment is unentangled since we can imagine a scenario where a system and some part of its environment are entangled before one applies a map. In light of the eventual direction of this work, we will turn to considering a universe made up of four qubits, illustrated in the following figure: In Fig. 4, we consider

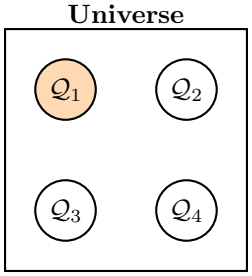

Figure 4: A universe made up of four qubits that will motivate our discussion.

our system to be $\mathcal{Q}_1$ and its environment to be $\mathcal{Q}_2, \mathcal{Q}_3$, and $\mathcal{Q}_4$. In deriving $\Lambda_S[\rho^{SE}_{(i)}]$, we made the assumption that the system and environment were initially not interacting and we could express the initial state as a product state. A unitary was then applied to act on and therefore entangle $\mathcal{Q}_1$ and $\mathcal{Q}_2$ and we arrived at the time evolution of the system from tracing over the effect of $\mathcal{Q}_2$ on $\mathcal{Q}_1$. The story changes if one of our newly-added qubits, $\mathcal{Q}_3$ or $\mathcal{Q}_4$, are initially interacting with $\mathcal{Q}_1$. It's clearly the case that $\mathcal{Q}_1$ is not entangled with $\mathcal{Q}_2, \mathcal{Q}_3$, or $\mathcal{Q}_4$ before U_1 is applied to $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Because our universe is made up of these four qubits which are initially unentangled, given U_1, we can derive $\Lambda_S[\rho^{SE}_{(i)}]$. Because there is no initial correlation between our system, $\mathcal{Q}_1$, and our environment, the rest of the three qubits, $\Lambda_S[\rho^{SE}_{(i)}]$ will always return a valid quantum state for $\rho^S(t)$ since our initial state for the entire circuit can be written as $\rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \rho_4$. If are only interested in describing the dynamics of $\mathcal{Q}_1$, there is initially no action on $\mathcal{Q}_3$ and $\mathcal{Q}_4$ so we may write this as

$$(\Lambda_{\mathcal{Q}_1} \otimes \mathbb{1}_{\mathcal{Q}_3} \otimes \mathbb{1}_{\mathcal{Q}_4})(\rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \rho_4) = \Lambda_1[\rho^{12}_{(i)}] \otimes \rho_{\mathcal{Q}_3} \otimes \rho_{\mathcal{Q}_4}. \tag{36}$$

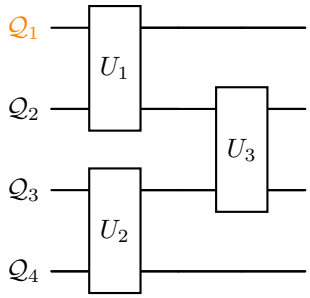

Figure 5: The brickwork circuit that will help us emphasize why we must remove the assumption that the system and environment are initially unentangled in order to derive dynamical maps.

However, looking at the second layer of this circuit, it becomes clear why we need to remove the restriction of the initial state of our system and environment be unentangled. Before U_3 is applied to Q_3 and Q_4, there is clearly entanglement between $\mathcal{Q}_2$ and $\mathcal{Q}_1$ due to the previous layer. If we therefore look to evolve $\mathcal{Q}_2$ in time, treating it as the system and the rest of the qubits as the environment, and we make the same assumption as before that we can write

$$(\Lambda_{\mathcal{Q}_2} \otimes \mathbb{1}_{\mathcal{Q}_1} \otimes \mathbb{1}_{\mathcal{Q}_4})(\rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \rho_4) = \Lambda_2[\rho^{23}_{(i)}] \otimes \rho_{\mathcal{Q}_1} \otimes \rho_{\mathcal{Q}_4}, \tag{37}$$

this mapping will not result in a valid output state since there is now actually some entanglement between the system and environment and we can't write the action of this map as a tensor product, $\Lambda_2[\rho^{23}_{(i)}] \otimes \rho_{\mathcal{Q}_1} \otimes \rho_{\mathcal{Q}_4}$. Because $\mathcal{Q}_1$ is entangled with $\mathcal{Q}_2$ before the unitary is applied to $\mathcal{Q}_2$ and $\mathcal{Q}_3$, the derived map $\Lambda_2[\rho^{23}_{(i)}]$ will *also* impact the state of $\mathcal{Q}_1$ (and $\mathcal{Q}_4$ for the same reason), meaning $\Lambda_2[\rho^{23}_{(i)}] \otimes \rho_{\mathcal{Q}_1} \otimes \rho_{\mathcal{Q}_4}$ will not result in a valid state.

We must therefore strengthen our definition of what it means for a map to return physical quantum states. If a system is embedded in a universe with d other subsystems, and upon applying a map Λ_S to the system and leaving all other subsystems unchanged, if $\Lambda_S \otimes \mathbb{1}^{(d)}$ for all d values, then the map is said to be **completely positive** (CP). If a map is not CP, there exists some entanglement between your system and at least one of your subsystems, which is why assuming a tensor product state will not always result in a valid output quantum state of your system after the map is applied. The map is called Not Completely Positive (NCP) map in this case. Given an NCP map, one can freely tweak the parameters that show up in the map to find conditions on them that do result in a valid quantum state. The range of parameter values of the mapping that result in valid quantum state outputs is called the Domain of Positivity (DOP). It follows that a CP map's DOP is all possible values of the map's parameter space.

2.3 Derivation of the Lindbladian

In the previous section, we considered the time evolution of the state of a system, represented as ρ^S, which openly interacted with a *closed* environment, ρ^E. We represented the composite of the system and environment with a tensor product, $\rho^{SE} = \rho^S \otimes \rho_E$. Since we specified that the full system and environment was closed, we were allowed to assume that ρ^{SE} would evolve unitarily and we could then specifically describe the state and its time evolution of the system by tracing out the environment; $\rho^S = \text{tr}_E(\rho^{SE})$.

All of this had assumed that we were capable of knowing the entire density matrix of the environment ρ^E, which is not often a reasonable assumption given that a quantum system can have a large number of interactions. For this reason, it is desirable to arrive at a way of describing the time evolution of ρ^S individually, without assuming we know what ρ^E looks like. Since we are considering an open system, we know that it will *not evolve unitarily* so we must approach this problem from another direction.

To develop a differential equation for the time evolution of any system, one needs to assume that the state at $t+dt$ is entirely determined by its state at t. A system that satisfies this condition is called *Markovian*. In proceeding with developing a way of describing the time evolution of an open system, we would also like to relax the assumption of the quantum mechanical state undergoing a Markovian process. Just as when only considering an interacting system as opposed to the full system and environment we had to remove the assumption of unitary time evolution, removing the assumption of its evolution being Markovian is reasonable to consider. In fact, the fluctuation-dissipation theorem tell us that information leaving a system into its environment must always flow back, meaning the state of our system might be influenced by one of its previous states at some later time.

Despite these theoretical concerns, there are scenarios where a system can be treated as Markovian in practice by making a Markovian approximation. (**should detail this more here**)

Recall that the time evolution of canonical variables p and q can be determined by just taking the Poisson bracket of it with the Hamiltonian of the system, written as $H = \frac{p^2}{2m}$. The Poisson brackets are generally defined as

$$\{Q_1, Q_2\} := \frac{\partial Q_1}{\partial x}\frac{\partial Q_2}{\partial p} - \frac{\partial Q_1}{\partial p}\frac{\partial Q_2}{\partial x}. \tag{38}$$

so with straightforward computation, one finds that

$$\{x, H\} = \frac{\partial H}{\partial p} = \frac{\partial}{\partial p}\left(\frac{p^2}{2m} + U(x)\right) = \frac{p}{m} \equiv \frac{dx}{dt} \tag{39}$$

and

$$\{p, H\} = -\frac{\partial H}{\partial x} = \frac{\partial}{\partial x}\left(\frac{p^2}{2m} + U(x)\right) = -\frac{\partial U}{\partial x} \equiv F = \frac{dp}{dt}. \tag{40}$$

In general, we find that the Poisson bracket of any *state variable* Q, such as position and momentum, with the Hamiltonian allows us to describe its time evolution, $\{Q, H\} \equiv \frac{dQ}{dt}$.

It is for this reason that we would like to develop a generalized Hamiltonian that may be applied to open systems. With such an object, we could arrive at the time evolution of the state of an open in a similar fashion without needing to assume full knowledge of the environment, as was done in the previous section.

Recall that the differential equation

$$\frac{dv}{dt} = Av \tag{41}$$

with the initial condition $v(0) = 0$, which is an element of some vector space V and where A is some linear operator that takes $A : V \to V$ has the general solution

$$v(t) = e^{At}v(0) \tag{42}$$

where we can describe the exponetiation of a matrix by its Taylor series expansion, $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$.

In the Schrödinger equation, $V \subseteq \mathcal{H}$ and $A = -iH$. We then have the differential equation

$$\frac{dv}{dt} = -iHv \tag{43}$$

which has the general solution

$$v(t) = e^{-iHt}v(0) \tag{44}$$

This scheme works for density matrices as well, where A is instead the *superopertor* (a linear operator that acts on other linear operators... since ρ itself is a linear operator) $-i[H, \cdot]$, which maps $-i[H, \cdot]/ : \rho \to \rho$. We then have the differential equation

$$\frac{d\rho}{dt} = -i[H, \rho]\rho(0) \tag{45}$$

which has the solution

$$\rho(t) = e^{iHt}\rho(0)e^{-iHt} \tag{46}$$

For the case of an open system, we therefore want a superoperator which generalizes the Hamiltonian, called the *Lindbladian* $\mathcal{L}$, and maps density matrices to density matrices of the system. We are looking for a $\mathcal{L}$ in the differential equation

$$\frac{d\rho}{dt} = \mathcal{L}[\rho] \tag{47}$$

which would have the general solution

$$\rho(t) = e^{\mathcal{L}t}\rho(0). \tag{48}$$

Recall that every completely positive operator and superoperator $\mathcal{E}$ has the operator sum representation (Kraus form) given by

$$\rho(t + dt) = \mathcal{E}_{dt}(\rho(t)) = \sum_{\mu} \langle e_\mu | U_{SE} | e_0 \rangle \rho_S \langle e_\mu | U_{SE}^\dagger | e_0 \rangle \tag{49}$$

where $\{e_\mu\}$ is the orthonormal basis for $\mathcal{H}_E$, so $|e_0\rangle$ is the state of the environment before the interaction takes place. If we define $M_\mu := \langle e_\mu | U_{SE} | e_0 \rangle$, which represents the averaging over the effects of the environment as it transitions from $|e_0\rangle$ to $|e_\mu\rangle$, we have that every operator and superoperator can be expressed as

$$\mathcal{E}_{dt}(\rho(t)) = \sum_{\mu} M_\mu \rho_E M_\mu^\dagger. \tag{50}$$

We can therefore represent

$$\rho(t) = e^{\mathcal{L}t}\rho(0) = \sum_{\mu} M_\mu \rho_E M_\mu^\dagger. \tag{51}$$

We can then approximate $\dot{\rho}(t)$ by:

$$\dot{\rho}(t) = \lim_{dt\to 0} \frac{\rho(t+dt) - \rho(t)}{dt} \implies \rho(t+dt) = \rho(t) + \dot{\rho}dt \tag{52}$$

with the Markovian assumption.

This can again be represented in the operator-sum form:

$$\rho(t+dt) = \rho(t) + \dot{\rho}dt = \sum_{\mu} M_\mu \rho(t) M_\mu^\dagger. \tag{53}$$

Because we have $\rho(t) + \dot{\rho}dt = \sum_\mu M_\mu \rho(t) M_\mu^\dagger$, it must be the case that **I DON'T GET THIS ARGUMENT** M_μ be equal to the identity to zeroth order, be of order $\sqrt{dt}$, or both. Combining these assumptions with the idea that $M_\mu,\ \mu > 0$ will describe the quantum jumps the system may undergo with probability dt, we must have that

$$\begin{aligned} M_0 &= \mathbb{1} + O(dt) \\ M_\mu &= \sqrt{dt} L_\mu,\ \mu \in \mathbb{N}. \end{aligned} \tag{54}$$

I ALSO DON'T UNDERSTAND WHAT THE L'S ARE... where the L_μ terms are called jump operators. Since every matrix can be rewritten into a Hermitian and ant-Hermitian part, we can re-express M_0 as:

$$M_0 = \mathbb{1} + (-iH + K)dt \tag{55}$$

where H is Hermitian and K is anti-Hermitian **don't fully understand why this separation is being done other than for utility in the proof...**

We first use the normalization condition of the operator-sum representation,

$$\mathbb{1} = \sum_{\mu\geq 0} M_\mu^\dagger M_\mu, \tag{56}$$

to show that

$$\begin{aligned} \mathbb{1} &= [\mathbb{1} + (-iH + K)dt]\,[\mathbb{1} + (iH + K)dt] + dt \sum_{\mu>0} L_\mu^\dagger L_\mu \\ &= \mathbb{1} + 2Kdt + dt^2(-iH + K)(iH + K) + dt \sum_{\mu>0} L_\mu^\dagger L_\mu \end{aligned} \tag{57}$$

and since the $(-iH + K)(iH + K)$ is of order dt^2, we may approximate this to go to zero since dt is already a small number. We then have that:

$$0 = 2Kdt + dt \sum_{\mu>0} L_\mu^\dagger L_\mu \implies K = -\frac{1}{2}\sum_{\mu>0} L_\mu^\dagger L_\mu. \tag{58}$$

With this expression for K, we may now take a look at $M_0 \rho M_0^\dagger$ again:

$$\begin{aligned} M_0 \rho M_0^\dagger &= [\mathbb{1} + (-iH + K)dt]\,\rho\,[\mathbb{1} + (iH + K)dt] \\ &= [\mathbb{1} + (-iH + K)dt]\,[\rho + \rho iHdt + \rho Kdt]\,. \end{aligned} \tag{59}$$

Again assuming $\mathcal{O}(dt^2) \approx 0$, we get:

$$M_0 \rho M_0^\dagger = \rho + \rho iH - iH\rho + K\rho dt + \rho K dt. \tag{60}$$

Recalling the expression for the commutator, $[a, b] = ab - ba$, we can simplify this into:

$$M_0 \rho M_0^\dagger = \rho - i[H, \rho] + dt(K\rho + \rho K). \tag{61}$$

Note that when the system is isolated, the K term would go to zero and we would be left with the usual time evolution of a density matrix, the Von Neumann equation.

Since we now have an expression for $M_0\rho M_0^\dagger$ and we have defined $M_\mu = \sqrt{dt}L_\mu$ for $\mu > 0$, we can determine the full expression of $M_\mu\rho M_\mu^\dagger$ for $\mu \geq 0$:

$$\begin{aligned} M_\mu\rho M_\mu^\dagger &= \rho + dt\left[-i[H,\rho] + K\rho + \rho K + \sum_{\mu>0} L_\mu\rho L_\mu^\dagger\right] \\ &= \rho + dt\left[-i[H,\rho] + \sum_{\mu>0} L_\mu\rho L_\mu^\dagger - \frac{1}{2}L_\mu^\dagger L_\mu\rho - \frac{1}{2}\rho L_\mu^\dagger L_\mu\right]. \end{aligned} \tag{62}$$

Using the definition of the anti-commutator, we have:

$$M_\mu\rho M_\mu^\dagger = \rho + dt\left[-i[H,\rho] + \sum_{\mu>0} L_\mu\rho L_\mu^\dagger - \frac{1}{2}\{L_\mu^\dagger L_\mu,\rho\}\right]. \tag{63}$$

Finally, since we said that $\rho(t) + \dot{\rho}dt = \sum_\mu M_\mu\rho(t)M_\mu^\dagger$, we now have an expression for $\dot{\rho}$:

$$\boxed{\dot{\rho} \equiv \mathcal{L}[\rho] = -i[H,\rho] + \sum_{\mu>0} L_\mu\rho L_\mu^\dagger - \frac{1}{2}\{L_\mu^\dagger L_\mu,\rho\}}. \tag{64}$$

What we have derived is called the **Lindbladian**.

This is generalized to include decoherence rates γ_μ in the GKSL form:

$$\mathcal{L}[\rho] = -i[H,\rho] + \sum_{\mu>0}\gamma_\mu\left(L_\mu\rho L_\mu^\dagger - \frac{1}{2}\{L_\mu^\dagger L_\mu,\rho\}\right) \tag{65}$$

Key difference between Lindbladian and quantum channels: While the Lindbladian and quantum channels both describe the time evolution of a quantum state, the Lindbladian describes the continuous time evolution of the state with a differential equation while quantum channels describe the discrete time evolution of the state in terms of a mapping that has certain properties.

2.4 Non-Markovian Maps

There has been a recent increase of interest in developing *dynamical maps* for non-Markovian processes. Recall that when deriving the Linbladian, we made the assumption that the evolution of the quantum state was Markovian, meaning that the state $\rho(t + dt)$ only depended on $\rho(t)$ as opposed to also depending on $\rho(t - \epsilon)$ for some ϵ. In the previous section, we described why we know that this assumption of quantum state evolution as Markovian is not theoretically realistic. We know it is likely that some information that leaves a system in a closed environment will eventually re-enter it given enough time. Such a situation would violate the assumption of the system evolution being Markovian and is why developing non-Markovian maps is an important task, especially with the development of quantum computation and quantum information. This is illustrated in Fig. 6:

To develop a dynamical quantum channel that does not assume a Markovian process, one needs to introduce some simplifying assumptions, typically in the form of a symmetry. A **covariant quantum channel** is one of these channels.

Definition 2.1 (Covariant Quantum Channel)**.** Consider a finite group G along with its two unitary representations U_k on Hilbert spaces $\mathcal{H}_k, k = 1, 2$ and a completely positive (CP), trace-preserving map $\Lambda : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$. Λ is a covariant quantum channel with respect to U_1 and U_2 if

$$\Lambda[U_1(g)\rho U_1^\dagger(g)] = U_2(g)\Lambda[\rho]U_2^\dagger(g)\ \forall\ \rho \in \mathcal{B}(\mathcal{H}_1)\ \&\ g \in G. \tag{66}$$

Note that $\mathcal{B}(\mathcal{H}_1)$ represents bounded linear operators on the Hilbert space of system one (I'm pretty sure...).

An important example of covariant channels are called **phase-covariant** qubit maps, which are covariant with respect to the operator $U(\phi) = e^{-i\sigma_z\phi}\ \forall\ \phi \in \mathbb{R}$.

Connection between NCP maps and non-Markovianity: CP maps rely on the assumption of the underlying process being Markovian; meaning that the process is *memoryless* (that the current state of the system can only be effected by the immediately previous state of the system and not its state some other amount of time ago). When a process is non-Markovian, you must use NCP maps to describe the dynamics of the system.

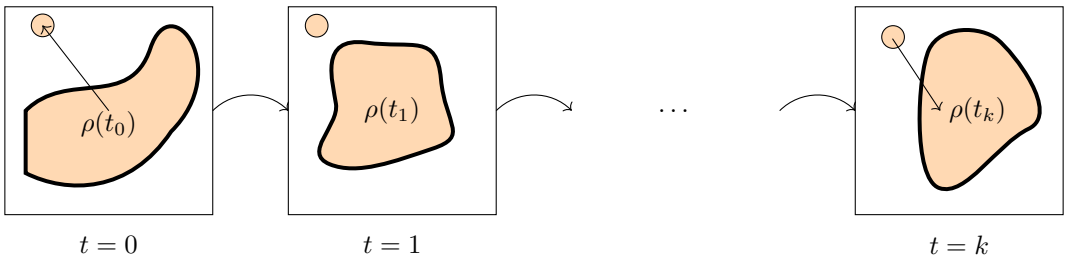

Figure 6: Information contained in the orange quantum system escapes to the environment at time $t = 0$. We state evolve the system using a regular dynamical quantum channel, such as the Linbladian defined above, which depends on the assumption that the state of the system $\rho(t + dt)$ only depends on its immediately previous state $\rho(t)$. The state evolves as a Markovian process for some time before this information re-enters the system at time $t = k$. Notice that the state of the system at $\rho(k)$ no longer depends on $\rho(k - dt)$; it also depends on the state of the system $\rho(t_0)$ when this information was released. This common occurrence in realistic quantum system demonstrates the necessity for the development of dynamic quantum channels that do not depend on the assumption of the evolution being Markovian.

Note that we also assumed that $\Lambda(0) = \mathbb{1}$ when developing dynamical maps that relied on the Markovian assumption, meaning that we assumed there was no initial correlation between the system and environment until $t > 0$ (this led us to develop a CP map). If this is not the case; that if the system and environment are truly always correlated and we can't apply this assumption, then we may need to use an NCP map to describe the temporal dynamics of the system.

Another violation of Complete Positivity can arise if a transform that's applied to the system is correlated with the state of the system.

3 Bipartite and Multipartite Entanglement

3.1 Bipartite Entanglement

3.2 Multipartite Entanglement