Considering the Influence of Exotic Structures on the Cosmic Microwave Background

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Abstract

(this is not my actual abstract) My thesis will be on adding modifications to the comic microwave background's angular power spectrum. There is lots of literature on going from a regular power spectrum $P(k)$ of the early universe's fluctuations to the observed C_ℓ spectrum that we can now observe well. The primordial fluctuations, captured by $P(k)$, are taken to be a power law characterized by an amplitude A_s and a scalar spectral index n_s. I will add effects to this $P(k)$ from variety of objects, such as random circles, random straight lines, and concentric circles. These added effects will come with a few parameters whose optimal values I will determine by minimizing the error between the resulting C_ℓ spectrum model, which I will calculate using the CAMB Python package, and observed CMB data from the Planck satellite (or I will use newer data if I can find some... Planck data is currently all that I have). Once this is done, I will make comments on the presence or absence (very likely the absence) of these added features in the universe and the scales that one would need to probe today in order to determine if they exist or not.

1 Introduction

2 Background Material

2.1 Correlation Functions and Power Spectra

Two point correlation function $\xi_2(\Delta\boldsymbol{r})$ between two points $\boldsymbol{r}$ and $\boldsymbol{r}'$ is defined as the expectation value of the field evaluated at two separate points;

$$\xi_2(\Delta\boldsymbol{r}) = \langle \phi(\boldsymbol{r})\phi(\boldsymbol{r} + \Delta\boldsymbol{r}) \rangle \,. \tag{1}$$

Given some point $\boldsymbol{r}_0$ in a field $\phi(\boldsymbol{r})$, the two-point correlation function tells you how likely you are to find the same field value at a separate point $\boldsymbol{r}_0+\Delta\boldsymbol{r}$. For a completely uncorrelated field, the correlation function becomes

$$\xi_2(|\boldsymbol{r}_i - \boldsymbol{r}_j|) = \langle \phi(\boldsymbol{r}_i)\phi(\boldsymbol{r}_j) \rangle = \delta_{ij},$$

in other words, knowing $\phi(\boldsymbol{r}_i)$ tells us no information about $\phi(\boldsymbol{r}_j)$, which is the definition of an uncorrelated, or random, field. .

In contrast, the two-point correlation function for a completely uniform field just returns the square of the field value,

$$\xi_2(|\boldsymbol{r}_i - \boldsymbol{r}_j|) = \langle \phi(\boldsymbol{r}_i)\phi(\boldsymbol{r}_j) \rangle = \phi_0^2,$$

since $\phi(\boldsymbol{r}_i) = \phi_0$ for all i.

The two-point correlation function, Eq. (1), can be defined using the expectation value's spatial integral, which gives us

$$\xi_2(\Delta\boldsymbol{r}) = \frac{1}{V}\int \phi(\boldsymbol{r})\phi(\boldsymbol{r} + \Delta\boldsymbol{r})\; d^3x. \tag{2}$$

If we want to describe the regularity of some field in terms of its spatial frequencies, we take the Fourier transform of Eq. (2) and use some tricks common in Fourier analysis to arrive at

$$\xi_2(\Delta \boldsymbol{r}) = \frac{1}{(2\pi)^3} \int P(k) e^{-i\boldsymbol{k}\cdot\Delta\boldsymbol{r}} d^3k. \tag{3}$$

If we are given a correlation function for a field and instead want to arrive at its power spectrum, we take the inverse of Eq. (3) to find

$$P(k) = \frac{1}{(2\pi)^3} \int \xi_2(\boldsymbol{r}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} d^3r \tag{4}$$

This tells us that the two-point correlation function $\xi_2(\Delta \boldsymbol{r})$ and the power spectrum $P(k)$ are Fourier pairs. The derivation of Eq. (3) is given in Appendix A.

2.2 Cosmological Physics and the Cosmic Microwave Background Radiation

The theory of cosmic inflation, the leading class of models that describe the very early phase of our universe's development[1], was originally posed in 1981 [1] as a modification to the standard big bang model, motivated by the standard theory's flaws that become relevant when extrapolating it back to very early times. Inflation posits that the early universe underwent a period of exponential expansion driven by the dynamics of a scalar field $\phi(\boldsymbol{r}, t)$, called the inflaton. Under inflation, the scale of the universe increased by an enormous amount, around 10^{50} times greater than what is predicted by standard cosmology [5].

Given the exponential expansion described by inflation, the inflaton field $\phi(\boldsymbol{x}, t)$ begins as a quantum field, subject to quantum fluctuations. Quantum fluctuations exist on imperceptible scales, but under the process of inflation, these primordial fluctuations were stretched astronomical scales, eventually being "frozen in" as classical density fluctuations $\delta\rho(\boldsymbol{x})$ which formed the seeds for cosmic structure.

[1] See [1, 2, 3, 4] for foundational papers in the theory of inflation.

The cosmic microwave background radiation (CMB), the first light released from the universe was first serendipitously discovered in 1964 by Arno Penzias and Robert Wilson [6]. Many telescopes and observational facilities have since been built to precisely measure this radiation, with the first, called the Cosmic Microwave Background Explorer (COBE), telescope launched in 1989 [7]. With these measurements, with small deviations from this mean slowly being discovered with advancements in observational , as Fig. 1

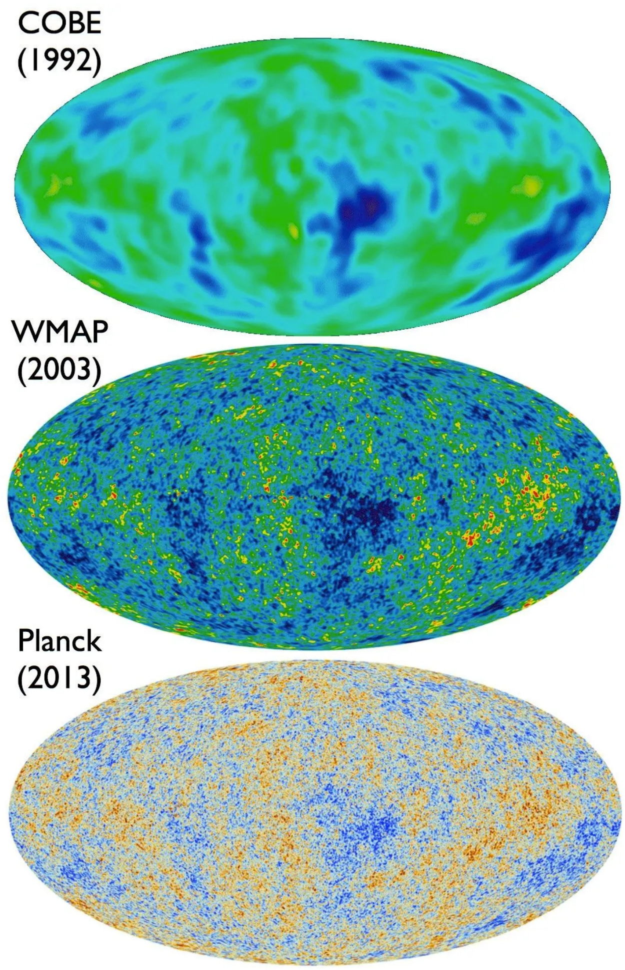

Figure 1: Cosmic microwave background temperature anisotropies as measured by COBE in 1990 [8], WMAP in 2003 [9], and the Planck satellite in 2018 [10] demonstrate the slow emergence of anisotropies from $T = 2.72548 \pm 0.00057K$.

As we have more confidently probed the CMB, we have discovered there to small deviations away from the mean temperature (quantify these variations here), caused by the primordial fluctuations. These anisotropies are ultimately led to the formation of large-scale structure and are therefore responsible for the development of life.

Modeling the CMB is vital to understanding the early stages of the universe, as this is the only current observational evidence of this stage[2]. To model the CMB, it is standard in cosmology to plot the angular power spectrum of the perturbations, denoted $C(\ell)$, where the multipole moment ℓ is the angular analog of wavenumber k, since the CMB is ultimately a three-dimensional field.

Given a power spectrum $P(k)$, one can project this onto a angular spectrum using the relation

$$C_\ell = \frac{4\pi}{(2\pi)^3} \int P(k) j_\ell^2(kr)\ dk, \tag{5}$$

which is derived in Appendix B

In standard inflationary cosmology, the primordial power spectrum $\mathcal{P}(k)$ is modeled as a power law with a free amplitude A and spectral index n_s;

$$\mathcal{P}(k) = A_s \left(\frac{k}{k_0}\right)^{1-n_s}. \tag{6}$$

From current CMB observations, we have determined $A_s =$ and $n_s =$. Intuitively, this power law tells us that **(what? That there is no clearly defined structure?)**

2.3 Bayesian Statistics

(maybe I should add some motivation here? Or should I just define Bayes theorem?) Bayes theorem states that

$$P(\Theta|D) = \frac{P(\Theta)P(D|\Theta)}{P(D)}. \tag{7}$$

[2]Cite papers on primordial GWs. Kamionkowski's first paper.

Where $P(\Theta)$, called the *prior* distribution, represents our prior knowledge about the values that our parameters of interest Θ could take on. When we have no knowledge about our parameters Θ, a constant prior distribution is chosen since this equally weights all possible parameter values.

$P(D|\Theta)$ is called the *likelihood*, which represents the probability of generating the observed data D given model parameters Θ and is often written as $\mathcal{L}(\Theta)$[3].When measurements made to construct the data are made up of random, independent samples, the central limit theorem is used to state that the likelihood for these processes should be a Gaussian distribution;

$$\mathcal{L}(\Theta) \propto \exp\left(-\sum_i \frac{\big(D(x_i) - M(x_i)\big)^2}{2\sigma_i^2}\right), \tag{8}$$

$P(D)$ is called the *evidence*, which represents the probability of seeing the data under all possible parameter values;

$$P(D) = \int P(D|\Theta)P(\Theta)d\Theta. \tag{9}$$

The evidence acts as a normalizing factor and is typically ignored in techniques that sample from the likelihood and prior, such as Markov chain Monte Carlo (MCMC) techniques**(should I define this too or just cite resources on it?)**. Sampling techniques are especially useful when the model parameters make up a high dimensional space, which makes the integral in Eq. (9) very computationally expensive.

The result of Eq. (7), the posterior distribution $P(\Theta|D)$ tells us the probability parameter values explain the given data. Achieving a value distribution for each parameter is the goal of Bayesian statistics, for this tells us which parameter values most likely fit the given data. When the posterior cannot be directly evaluated, it is often approximated using MCMC sampling.

(explain what the Bayes factor is and relate it Occam's razor)

[3]It's crucial to state that the likelihood is *not* a probability distribution for parameter values of Θ. It is a function $f(\Theta)$ that quantifies how well the parameters describe the observed data.

$$K = \frac{P(\Theta_{\text{mod}}|D)}{P(\Theta_{\Lambda CDM}|D)} \frac{P(\Theta_{\Lambda CDM})}{P(\Theta_{\text{mod}})} \tag{10}$$

3 Procedures

This thesis will investigate the influence of additional structure on the primordial power spectrum by comparing the resulting C_ℓ spectrum that's generated with observational data from the Planck satellite, which is openly provided by the Planck Legacy Archive. The two-point correlation function has been derived for a variety of random fields, such as those of lines and concentric circles in [11]. Given these correlation functions, I will first derive their corresponding $P(k)$. I then construct a total modified primordial power spectrum $P'(k) = P(k) + \mathcal{P}(k)$ where $\mathcal{P}(k)$ is the standard primordial power spectrum given in Eq. (6), and use the Code for Anisotropies in the Microwave Background (CAMB) Python package [12] to generate resulting modified angular power spectra C'_ℓ. Given the set of modified angular power spectra that result from this procedure for each added $\xi_2(\Delta r)$, I then plot these against the spectrum generated by standard power-law inflationary cosmology in regards to their fit with Planck's data.

To determine the goodness-of-fit for these modifications, I first carry out a Bayesian analysis on the free parameters of each specific model to find their optimal values. With these, I then compute a Bayes factor with the power law data to quantify how likely the modified models explain the observed CMB data.

To perform Bayesian analysis with the goal of finding the optimal values for the modified spectrum parameters, which are the density of circles ρ and their radii R in the case of the identical circle field, we assume uniform priors on both of these parameters over a range **(I need to justify why the range I'm using is good)** since we have no prior knowledge of what value these should take. We also assume the likelihood follows the exponential distribution given in Eq. (8) because CMB measurements are taken randomly and independently.

Because we will evaluate the likelihood and priors computationally in Python, we can use a MCMC sampler rather than additionally computing the evidence integral, as explained in the background section on Bayesian statistics.

To determine how likely these modifications to the primordial power spectrum better explain the angular power spectrum of the CMB, I will finally compute the Bayes factor, as defined in the Background Material section. **(will probably be doing this without any Python packages. Haven't gotten to this point yet, though, so this will be added soon)**

3.1 Random Circle Field

For example, take the two-point correlation function for a random field of identical circles as derived in [11], given as

$$\xi_2(r) = \frac{1}{\pi \rho r \sqrt{1 - \left(\frac{r}{2R}\right)^2}} \tag{11}$$

where ρ is the density of these circles and R is their fixed radius. Using Eq. (4) to compute the power spectrum of this field, we find that

$$P_\circ(k) = \frac{1}{(2\pi)^3} \int_0^{2R} \frac{e^{ikr}}{\pi \rho r \sqrt{1 - \left(\frac{r}{2R}\right)^2}} dr = \frac{R}{2\pi k \rho} J_0(kR)^2 \tag{12}$$

where the integral's bounds come from the range of validity for the correlation function to hold.

The identical circle field power spectrum plotted on the scale relevant for the CMB is compared to the standard power law spectrum in Fig. 2.

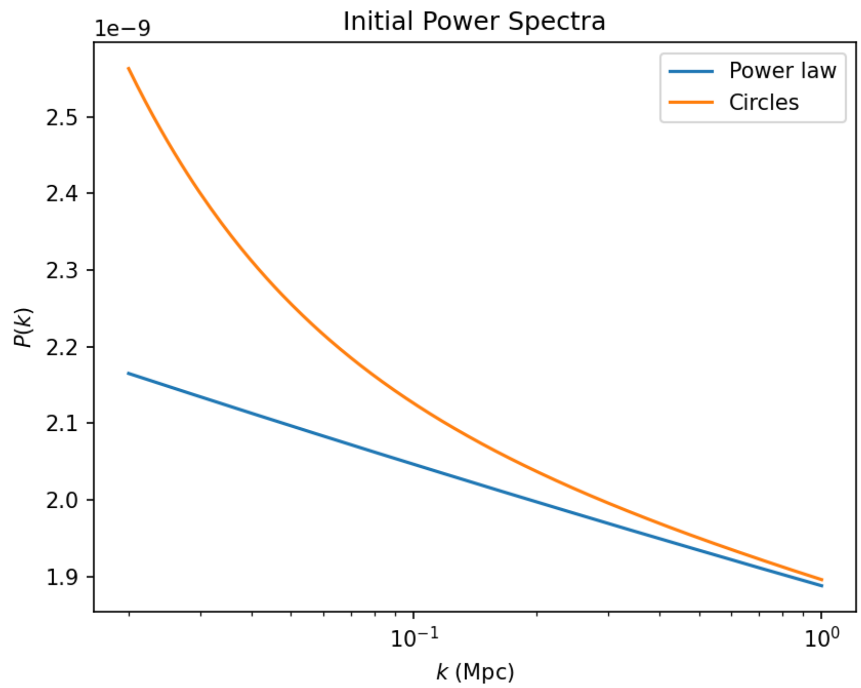

Figure 2: The power spectrum generated by the field of random circles compared against the typical power law spectrum.

To implement this modification into the CAMB code, we write the full power spectrum as a sum of the standard power law and the result of Eq. (12):

$$P'(k) = \mathcal{P}(k) + \frac{R}{2\pi k\rho} J_0(kR)^2. \tag{13}$$

3.2 Concentric Circle Field

... I will add a procedure similar to the one for random circles for each modification I eventually add (will derive the $P(k)$ for each added object in these sections)

4 Results

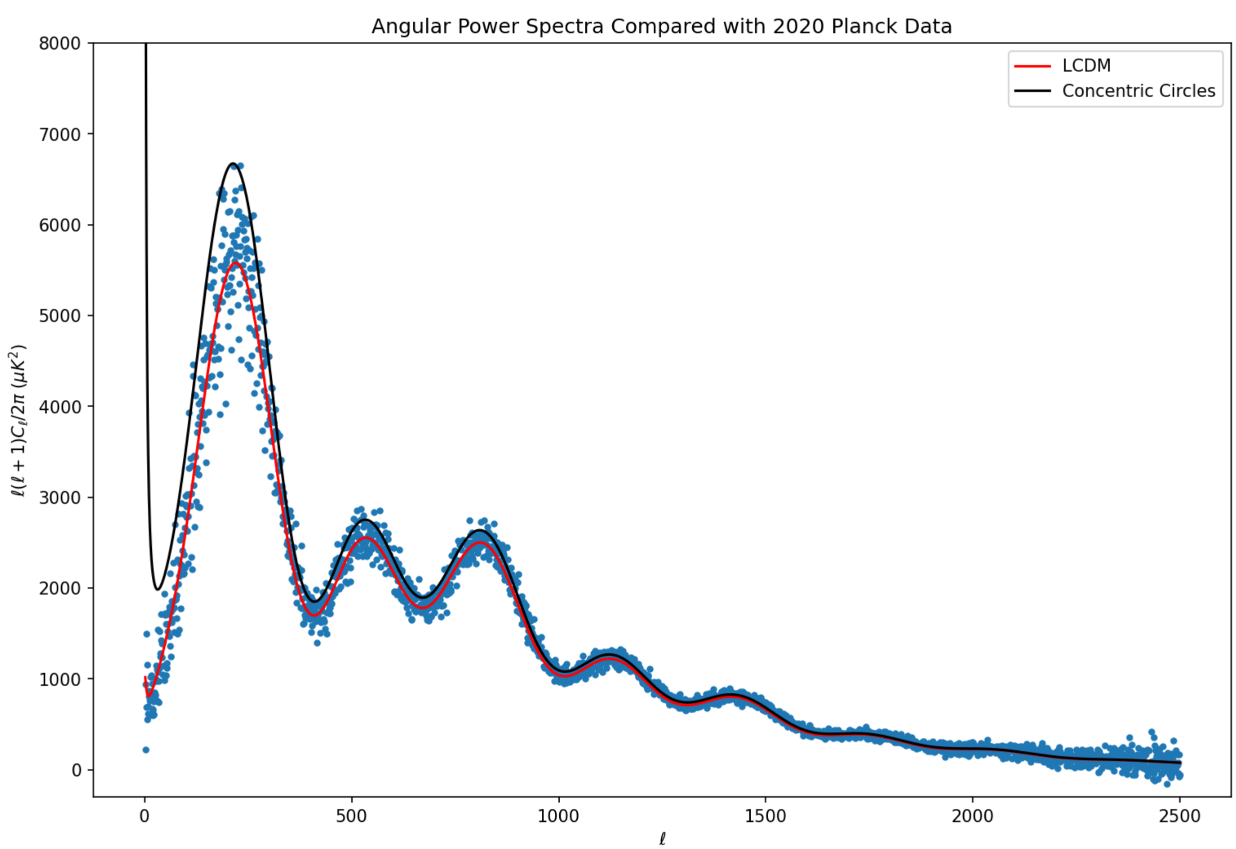

Figure 3: Model with nonideal parameters

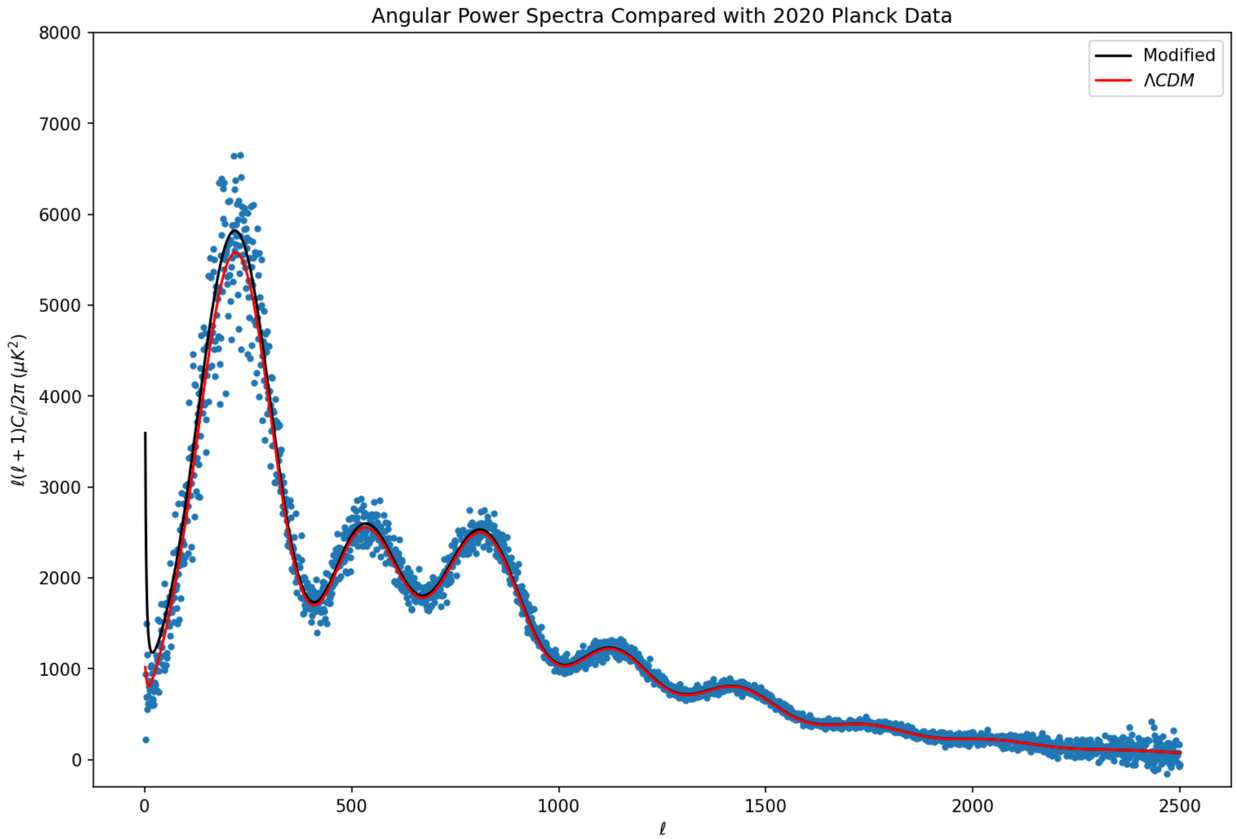

Figure 4: Caption

Adds odds ratio. Calculates how much more likely is an additional model better than the standard model. Typically want Bayes factor/odds ratio of around 8 (that it would be 8 times more likely that the modifications better explain the data than the standard model)

5 Discussion

... Probably not likely that any of these modifications better explain the data than standard cosmology. Explain using the computed Bayes factor.

6 Conclusion

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Based on observations obtained with Planck (**??**), an ESA science mission with instruments and contributions directly funded by ESA Member States, NASA, and Canada.

A Fourier Transform of Two-Point Correlation Function:

Take Fourier transform

$$\phi(\boldsymbol{r}) = \frac{1}{(2\pi)^3} \int \phi(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} d^3k \tag{14}$$

So if we define $\boldsymbol{r}' = \boldsymbol{r} + \Delta\boldsymbol{r}$, we get

$$\xi_2(\Delta\boldsymbol{r}) = \langle \phi(\boldsymbol{r})\phi(\boldsymbol{r}+\Delta\boldsymbol{r})\rangle = \left\langle \frac{1}{(2\pi)^3} \int \phi(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} d^3k \frac{1}{(2\pi)^3} \int \phi(\boldsymbol{k}') e^{-i\boldsymbol{k}'\cdot(\boldsymbol{r}+\Delta\boldsymbol{r})} d^3k' \right\rangle \tag{15}$$

using the fact that $\xi_2(\Delta\boldsymbol{r})$ is a real-valued function, which requires the complex conjugate to be taken of the $\mathscr{F}\{\phi(\boldsymbol{r}+\Delta\boldsymbol{r})\}$ term.

Using the Wiener-Khinchin Theorem for random variables in 3D, $\langle \phi(\boldsymbol{k})\phi(\boldsymbol{k}')\rangle = (2\pi)^3 \delta^3(\boldsymbol{k}-\boldsymbol{k}')P(k)$ [13], we get

$$\xi_2(\Delta\boldsymbol{r}) = \frac{1}{(2\pi)^3} \int\int \delta^3(\boldsymbol{k}-\boldsymbol{k}')P(k) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} e^{-i\boldsymbol{k}'\cdot(\boldsymbol{r}+\Delta\boldsymbol{r})} d^3k' d^3k. \tag{16}$$

Integrating over the $\delta^3(\boldsymbol{k}-\boldsymbol{k}')$ term results in

$$\xi_2(\Delta\boldsymbol{r}) = \frac{1}{(2\pi)^3} \int P(k) e^{-i\boldsymbol{k}\cdot\Delta\boldsymbol{r}} d^3k \tag{17}$$

B Deriving the Angular Power Spectrum from Initial Fluctuations

$$a_{\ell m} = \int \phi(\boldsymbol{r}) Y^*_{\ell m}(\hat{n}) d\Omega \tag{18}$$

Using the Fourier transform of $\phi(\boldsymbol{r})$

$$a_{\ell m} = \frac{1}{(2\pi)^3} \int\int \phi(\boldsymbol{k}) Y^*_{\ell m}(\hat{n}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\ d^3k\ d\Omega \tag{19}$$

Using the plane wave expansion:

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^\ell j_\ell(\boldsymbol{k}\cdot\boldsymbol{r}) Y_{\ell m}(\hat{\boldsymbol{r}}) Y^*_{\ell m}(\hat{\boldsymbol{k}}) \tag{20}$$

where the spherical Bessel function $j_\ell(kr)$ is defined as:

$$j_\ell(kr) = \sqrt{\frac{\pi}{2kr}} J_{\ell+1/2}(kr) \tag{21}$$

$$a_{\ell m} = \frac{4\pi}{(2\pi)^3} \int\int \phi(\boldsymbol{k}) Y^*_{\ell m}(\hat{n}) \sum_{\ell'=0}^{\infty} \sum_{m=-\ell'}^{\ell'} i^{\ell'} j_{\ell'}(\boldsymbol{k}\cdot\boldsymbol{r}) Y_{\ell' m'}(\hat{\boldsymbol{r}}) Y^*_{\ell' m'}(\hat{\boldsymbol{k}})\ d^3k\ d\Omega \tag{22}$$

Using the fact that

$$\int Y^*_{\ell m}(\hat{n}) Y_{\ell' m'}(\hat{n}) d\Omega = \delta_{\ell\ell'}\delta_{mm'}, \tag{23}$$

this gives us

$$a_{\ell m} = \frac{4\pi}{(2\pi)^3} \int \phi(\boldsymbol{k}) \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^\ell j_\ell(\boldsymbol{k}\cdot\boldsymbol{r}) Y^*_{\ell m}(\hat{\boldsymbol{k}})\ d^3k \tag{24}$$

Now take $\langle |a_{\ell m}|^2 \rangle$, using $\langle \phi(\boldsymbol{k})\phi(\boldsymbol{k}') \rangle = (2\pi)^3 \delta^3(\boldsymbol{k}-\boldsymbol{k}')P(k)$:

$$\left\langle |a_{\ell m}|^2 \right\rangle = \frac{4\pi}{(2\pi)^3} \int\int P(k)\delta^3(\boldsymbol{k}-\boldsymbol{k}') \sum_{\ell\ell'} \sum_{mm'} i^\ell j_\ell(\boldsymbol{k}\cdot\boldsymbol{r}) Y^*_{\ell m}(\hat{\boldsymbol{k}}) i^{\ell'} j_{\ell'}(\boldsymbol{k}'\cdot\boldsymbol{r}) Y_{\ell' m'}(\hat{\boldsymbol{k}'})\ d^3k' d^3k \tag{25}$$

$$\left\langle |a_{\ell m}|^2 \right\rangle = \frac{4\pi}{(2\pi)^3} \int P(k) \sum_{\ell\ell'} \sum_{mm'} i^\ell j_\ell(\boldsymbol{k}\cdot\boldsymbol{r}) Y^*_{\ell m}(\hat{\boldsymbol{k}}) i^{\ell'} j_{\ell'}(\boldsymbol{k}\cdot\boldsymbol{r}) Y_{\ell' m'}(\hat{\boldsymbol{k}})\ d\Omega dk \tag{26}$$

$$\left\langle |a_{\ell m}|^2 \right\rangle = \frac{4\pi}{(2\pi)^3} \int P(k) \sum_{\ell\ell'} \sum_{mm'} i^\ell j_\ell(\boldsymbol{k}\cdot\boldsymbol{r}) i^{\ell'} j_{\ell'}(\boldsymbol{k}\cdot\boldsymbol{r}) \delta_{\ell\ell'}\delta_{mm'}\ dk \tag{27}$$

$$\left\langle |a_{\ell m}|^2 \right\rangle = \frac{4\pi}{(2\pi)^3} \int P(k) \sum_{\ell} \sum_{m} i^{2\ell} j^2_\ell(\boldsymbol{k}\cdot\boldsymbol{r})\ dk \tag{28}$$

Since $i^{2\ell} = 1$,

$$C_\ell = \frac{4\pi}{(2\pi)^3} \int P(k) j^2_\ell(kr)\ dk \tag{29}$$

This tells us that given a power spectrum $P(k)$, we can generate an angular power spectrum $C(\ell)$ using this integral!