Quantum Field Theory for the Gifted Amateur - Selected Solutions

Andrew Valentini

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1 Chapter 1: Lagrangians

Exercise (1.5): For a three-dimensional elastic medium, the potential energy is

$$V = \frac{\mathcal{T}}{2} \int d^3x \, (\nabla\psi)^2 \, ,$$

and the kinetic energy is

$$T = \frac{\rho}{2} \int d^3x \left(\frac{\partial\psi}{\partial t}\right)^2 .$$

Use these results and show the functional derivative approach to show that ψ obeys the wave equation:

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2},$$

where v is the velocity of the wave.

Solution: Beginning by constructing the Lagrangian for this system, one finds that:

$$L = \int d^3x \frac{\mathcal{T}}{2} (\nabla\psi)^2 - \frac{\rho}{2}\left(\frac{\partial\psi}{\partial t}\right)^2 \tag{1}$$

which means that the Lagrangian density is:

$$\mathcal{L} = \frac{\mathcal{T}}{2} (\nabla\psi)^2 - \frac{\rho}{2}\left(\frac{\partial\psi}{\partial t}\right)^2 . \tag{2}$$

Applying Hamilton's principle of least action to this system:

$$\frac{\delta S}{\delta\psi} = 0 \implies \frac{\partial\mathcal{L}}{\partial\psi} - \partial_b \frac{\partial\mathcal{L}}{\partial(\partial_b\psi)} = 0 = \frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)} - \sum_{i=1}^{3}\frac{\partial}{\partial x^i}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x^i)} = 0 \tag{3}$$

Where $x^1 = x, x^2 = y$, and $x^3 = z$. Because $\mathcal{L}$ has no dependence on a ψ term, this functional derivative becomes:

$$-\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)} = \sum_{i=1}^{3}\frac{\partial}{\partial x^i}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x^i)} . \tag{4}$$

Taking these partial derivative of the Lagrangian density results in:

$$\begin{gathered} -\frac{\partial}{\partial t}\left(-\rho\frac{\partial\psi}{\partial t}\right) = \sum_{i=1}^{3}\frac{\partial}{\partial x^i}\left(\mathcal{T}\frac{\partial\psi}{\partial x^i}\right) \\ \implies \rho\frac{\partial^2\psi}{\partial t^2} = \mathcal{T}\nabla^2\psi \implies \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = \nabla^2\psi \end{gathered} \tag{5}$$

when $v = \sqrt{\mathcal{T}/\rho}$.

2 Chapter 2: Simple harmonic oscillators

Exercise (2.1): For the one-dimensional harmonic oscillator, show that with creation and annihilation operators defined as in Eq. (2.9) and Eq. (2.10), $[\hat{a}, \hat{a}] = 0$, $\left[\hat{a}^\dagger, \hat{a}^\dagger\right] = 0$, $\left[\hat{a}, \hat{a}^\dagger\right] = 1$, and $\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)$.

Solution: With the definitions:

$$\begin{aligned} \hat{a} &= \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \\ \hat{a}^\dagger &= \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega}\hat{p}\right), \end{aligned} \tag{6}$$

$[\hat{a}, \hat{a}]$ will be defined as:

$$[\hat{a}, \hat{a}] = \frac{m\omega}{2\hbar} \left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) - \frac{m\omega}{2\hbar} \left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) = 0 \tag{7}$$

and same logic will apply to $[\hat{a}^\dagger, \hat{a}^\dagger]$:

$$[\hat{a}^\dagger, \hat{a}^\dagger] = \frac{m\omega}{2\hbar} \left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) \left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) - \frac{m\omega}{2\hbar} \left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) \left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) = 0. \tag{8}$$

Now for $[\hat{a}, \hat{a}^\dagger]$ (using the fact that $[\hat{x}, \hat{p}] = i\hbar$):

$$\begin{aligned} \left[\hat{a}, \hat{a}^\dagger\right] &= \frac{m\omega}{2\hbar} \left[\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) - \left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) \left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)\right] \\ &\implies \frac{m\omega}{2\hbar} \left[\frac{-i}{m\omega}[\hat{x}, \hat{p}] + \frac{i}{m\omega}[\hat{p}, \hat{x}]\right] = \frac{m\omega}{2\hbar} \left[\frac{\hbar}{m\omega} + \frac{\hbar}{m\omega}\right] = 1 \end{aligned} \tag{9}$$

For the Hamiltonian, first invert the definitions of $\hat{a}$ and $\hat{a}^\dagger$ to find that:

$$\begin{aligned} \hat{x} &= \sqrt{\frac{\hbar}{2m\omega}} \left(\hat{a} + \hat{a}^\dagger\right) \\ \hat{p} &= -i\sqrt{\frac{\hbar m\omega}{2}} \left(\hat{a} - \hat{a}^\dagger\right) \end{aligned} \tag{10}$$

and plug these forms into the expression for the Hamiltonian:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \frac{-\omega\hbar}{4} \left(\hat{a} - \hat{a}^\dagger\right)^2 + \frac{\omega\hbar}{4} \left(\hat{a} + \hat{a}^\dagger\right) = \frac{\omega\hbar}{2} \left(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right) \tag{11}$$

and use the previously derived commutation relation $\left[\hat{a}, \hat{a}^\dagger\right] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$:

$$\hat{H} = \frac{\omega\hbar}{2} \left(\hat{a}^\dagger\hat{a} + 1 + \hat{a}^\dagger\hat{a}\right) = \omega\hbar \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \tag{12}$$

3 Chapter 3: Occupation number representation

Exercise (3.1): For boson operators satisfying

$$[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}] = \delta_{\boldsymbol{pq}}$$

show that

$$\frac{1}{\mathcal{V}} \sum_{\boldsymbol{pq}} e^{i(\boldsymbol{p}\cdot\boldsymbol{x} - \boldsymbol{q}\cdot\boldsymbol{y})} [\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}] = \delta^3(\boldsymbol{x} - \boldsymbol{y})$$

where $\mathcal{V}$ is the volume of space over which the system is defined. Repeat this for fermion commutation operators.

Solution: We compute

$$\begin{aligned} \frac{1}{\mathcal{V}} \sum_{\boldsymbol{pq}} e^{i(\boldsymbol{p}\cdot\boldsymbol{x} - \boldsymbol{q}\cdot\boldsymbol{y})} [\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}] &= \frac{1}{\mathcal{V}} \sum_{\boldsymbol{pq}} e^{i(\boldsymbol{p}\cdot\boldsymbol{x} - \boldsymbol{q}\cdot\boldsymbol{y})} \delta_{\boldsymbol{pq}} \\ &= \frac{1}{\mathcal{V}} \sum_{\boldsymbol{p}} e^{i\boldsymbol{p}(\boldsymbol{x} - \boldsymbol{y})} := \delta^3(\boldsymbol{x} - \boldsymbol{y}) \end{aligned} \tag{13}$$

where we are allowed to make the final step because we have written out the definition of the three-dimensional inverse Fourier transform in terms of the Bessel function.

If we then looked at

$$\frac{1}{\mathcal{V}}\sum_{\boldsymbol{pq}} e^{i(\boldsymbol{p}\cdot\boldsymbol{x}-\boldsymbol{q}\cdot\boldsymbol{y})}\{\hat{c}_{\boldsymbol{p}}, \hat{c}^\dagger_{\boldsymbol{q}}\} \tag{14}$$

for the fermion commutation relation, we would get the same result.

4 Chapter 4: Making second quantization work

5 Chapter 5: Continuous systems

6 Chapter 6: A first stab at relativistic quantum mechanics

Exercise (6.1): This problem looks forward to ideas which we will consider in more detail in the next chapter. Show that the Lagrangian density given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2,$$

yields the Klein-Gordon equation when using the Euler-Lagrange equation. Also derive expressions for the momentum $\pi = \partial\mathcal{L}/\partial\dot{\phi}$ and the Hamiltonian density $\mathcal{H} = \pi\dot{\phi} - \mathcal{L}$ for this Lagrangian and interpret the result.

Solution: To work out the equations of motion for this field, we compute

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi \tag{15}$$

and

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} &= \frac{\partial}{\partial(\partial_\mu\phi)}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right) = \frac{\partial}{\partial(\partial_\mu\phi)}\left(\frac{1}{2}\partial_\mu\phi\eta^{\mu\nu}\partial_\nu\phi\right)\\ &= \frac{1}{2}\eta^{\mu\nu}\partial_\nu\phi\frac{\partial}{\partial(\partial_\mu\phi)}(\partial_\mu\phi) + \frac{1}{2}\partial_\mu\phi\eta^{\mu\nu}\frac{\partial}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\\ &= \frac{1}{2}\eta^{\mu\nu}\partial_\nu\phi + \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\delta^\mu_\nu = \frac{1}{2}\partial^\mu\phi + \frac{1}{2}\eta^{\mu\nu}\partial_\nu = \partial^\mu\phi.\end{aligned} \tag{16}$$

Plugging these results into the Euler-Lagrange equations, we get

$$\left(m^2 + \partial_\mu\partial^\mu\right)\phi = 0 \tag{17}$$

which is the Klein-Gordon equation.

We then compute

$$\pi := \frac{\partial\mathcal{L}}{\partial\dot{\phi}} \equiv \frac{\partial\mathcal{L}}{\partial_0\phi} = \frac{\partial}{\partial_0\phi}\left(\frac{1}{2}(\partial_0\phi)^2 - (\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right) = \partial_0\phi \tag{18}$$

where the negative sign on the $(\nabla\phi)^2$ term came from the fact that we are working with the metric signature $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. Then we have that

$$\mathcal{H} = \partial_0^2\phi - \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\partial_0^2\phi + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2. \tag{19}$$

The $\partial_0^2\phi$ tells us that the change of ϕ over time contributes to the energy density of the field, which is related to the kinetic energy of the field. The $(\nabla\phi)^2$ term tells us that the spatial variations of ϕ contributes to the field's energy density, which could be related to the potential energy of the field. The $m^2\phi^2$ term tells us that the mass of the field's quanta (its particles) contributes to the field's energy density. It's interesting to note the difference in signage between these terms; the terms with a positive sign are the terms that seem to be properties inherent to the field while the term with the opposite sign seems to be related to particle creation. It would make sense for the properties inherent to the field to positively contribute to the field's energy density while the property responsible for particle creation out of this field would negatively contribute to the energy density of the field, since it likely requires energy from the field to create particles.

7 Chapter 7: Examples of Lagrangians, or how to write down a theory

Exercise (7.1): For the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \sum_{n=1}^{\infty}\lambda_n\phi^{2n+2}$$

show that the equation of motion is given by

$$(\partial^2 + m^2)\phi + \sum_{n=1}^{\infty}\lambda_n(2n+2)\phi^{2n+1} = 0$$

To derive the equation of motion for this field, we compute

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi - \sum_{n=1}^{\infty}\lambda_n(2n+2)\phi^{2n+1} \tag{20}$$

and

$$\frac{\partial}{\partial(\partial_\mu\phi)} = \partial_\mu^2 \tag{21}$$

so we have that

$$\mathcal{L} = \frac{1}{2}\partial_\mu\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \sum_{n=1}^{\infty}\lambda_n\phi^{2n+2}. \tag{22}$$

Exercise (7.2): Consider a massive scalar field $\phi(x)$ coupled to a source $J(x)$, described by the Lagrangian of eqn 7.10. Show that the equations of motion are those of eqn 7.11.

Solution:

Exercise (7.4): Show that eqn 7.7 is equivalent to

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2.$$

Show that

$$\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}$$

and

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2. \tag{23}$$

Define the quantity

$$\Pi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \tag{24}$$

and show that $\Pi^\mu = \partial^\mu$ and $\Pi^0 = \pi = \dot{\phi}$.

Solution: