Quant Finance Notes

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Contents

1 Mathematical Preliminaries

Definition 1. if x is a continuous random variable, then the **expected value** is

$$\mathbb{E}[x] = \int_{\mathbb{R}} x f(x) dx \tag{1}$$

where $f(x)$ is the probability density function (PDF) of x.

Definition 2. The **cumulative density function** (CDF) is defined as

$$\Phi(\alpha) := \text{Prob}(x \leq \alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-x^2/2} dx, \tag{2}$$

if $x \sim \mathcal{N}(0,1)$ since $f(x) = \exp(-x^2/2)/\sqrt{2\pi}$ is the PDF of $\mathcal{N}(0,1)$. The CDF defines the total probability up to a point α.

Example 1. $\mathbb{E}[x]$ for $x \sim \mathcal{N}(0,1)$:

$$\mathbb{E}[x] = \frac{1}{\sqrt{2\pi}} \int_{-\inf}^{\infty} e^{-x^2/2} x dx = 0 \tag{3}$$

since x is an odd function, $\exp(-x^2)$ is an even function, and the bounds are symmetric.

Example 2. $\mathbb{E}[x]$ for $x \sim \mathcal{N}(\mu, \sigma^2)$:

$$\mathbb{E}[x] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} x dx \tag{4}$$

define $u = \frac{x-\mu}{\sigma} \implies dx = \sigma du$,

$$\begin{aligned} \mathbb{E}[x] &= \frac{1}{\sqrt{2\pi}\sigma} \left[\sigma \int_{-\infty}^{\infty} e^{-u^2/2} u du + \sigma\mu \int_{-\infty}^{\infty} e^{-u^2/2} du \right] \\ &= \frac{\sigma\mu\sqrt{2\pi}}{\sqrt{2\pi}\sigma} = \mu. \end{aligned} \tag{5}$$

Definition 3. variance

Definition 4. Central limit theorem: if $x_1, x_2, \ldots, x_n$ is a sequence of identical but independent random variables, $x_i \sim X$, $\mathbb{E}[X] = \mu < \infty$, $\text{var}(X) = \sigma^2 < \infty$, and we define

$$y_n := \frac{x_1 + \cdots + x_n - n\mu}{\sigma\sqrt{n}} \tag{6}$$

then

$$\lim_{n \to \infty} y_n \sim \mathcal{N}(0, 1). \tag{7}$$

2 Stock Path Modeling

2.1 $n-$Step Risk-Free Stock Return Model

The most foundational idea involved in modeling the paths of the stock market is the $n-$**step random walk**. It assumes the daily fluctuations of the stock price are independent, random events which either drive the price up $+1$ or down -1, describing stocks as stochastic processes.

First simplifying this to the 1-step random walk, we consider a process governed by

$$w \sim \begin{cases} +1 & p = \frac{1}{2} \\ -1 & p = \frac{1}{2}. \end{cases} \tag{8}$$

Notice that $\mathbb{E}[w] = 0$ and $\sigma[w] = \mathbb{E}[w]^2 - \mathbb{E}[w^2] = 1$. An $n-$step random walk is then the chain of n 1-step random walks, $m_n = w_1 + w_2 + \cdots + w_n$.

Definition 5. First define the value of a stock at a given time to be S_t, which implies that S_{t_2}/s_{t_1} represents the return of the stock from time t_1 to t_2. The **1-step risk-free stock return model** is defined as

$$\frac{S_{t_2}}{S_{t_1}} = \begin{cases} 1+\sigma & p = \frac{1}{2} \\ 1-\sigma & p = \frac{1}{2} \end{cases} \tag{9}$$

with $\sigma > 0$ is the volatility of the stock over the *single time step.*

We compute

$$\begin{aligned} \mathbb{E}\left[\frac{S_{t_2}}{S_{t_1}}\right] =& \frac{1}{2}(1+\sigma) + \frac{1}{2}(1-\sigma) = 1 \\ \& \ \sigma\left[\frac{S_{t_2}}{S_{t_1}}\right] &= \sigma^2. \end{aligned} \tag{10}$$

If we wish to extend this model to a specific time interval $t_1 \to t_2$,

$$\frac{S_{t_2}}{S_{t_1}} = \begin{cases} 1+\sigma\sqrt{t_2 - t_1} & p = \frac{1}{2} \\ 1-\sigma\sqrt{t_2 - t_1} & p = \frac{1}{2} \end{cases} \tag{11}$$

where the square root is added for normalization.

If we extend this 1-step stock model to $n-$steps, we define an interval $[t_1, t_2] \equiv [0, t]$, which will feature n steps between the two endpoints. Our cumulative model return is then the multiplicative sum of the model return over each step:

$$\frac{S_t}{S_0} = \frac{S_{t/n}}{S_0}\frac{S_{2t/n}}{S_{t/n}} \cdots \frac{S_t/n}{S_{(n-1)t/n}} \equiv \prod_{i=1}^{n} \frac{S_{ti/n}}{S_{t(i-1)/n}} \tag{12}$$

where

$$\frac{S_{ti/n}}{S_{t(i-1)/n}} \sim \begin{cases} 1+\sigma\sqrt{t/n} & p=\frac{1}{2} \\ 1-\sigma\sqrt{t/n} & p=\frac{1}{2}. \end{cases} \tag{13}$$

When the limit $n \to \infty$ is taken (a continuous process), we arrive at the **geometric Brownian motion** model of stocks.

2.2 Geometric Brownian Motion

If you are interested in considering the *cumulative return* of a model, consider flipping a fair coin n times, where H_n indicates the number of heads and T_n indicates the number of tails ($H_n + T_n = n$). The $n-$step return model becomes

$$\frac{S_t}{S_0} = \left(1+\sigma\sqrt{t/n}\right)^{H_n}\left(1-\sigma\sqrt{t/n}\right)^{T_n} \tag{14}$$

If we take the log of both sides,

$$\ln\left(\frac{S_t}{S_0}\right) = H_n \ln\big(1+\sigma\sqrt{t/n}\big) + T_n \ln\big(1-\sigma\sqrt{t/n}\big). \tag{15}$$

Using the Taylor expansion for $\ln(1+x) \approx x - x^2/2 + x^3/3 - \ldots$,

$$\begin{aligned} \ln\left(\frac{S_t}{S_0}\right) &\approx H_n \ln\left(\sigma\sqrt{t/n} - \frac{\sigma^2 t}{2n}\right) + T_n \ln\left(-\sigma\sqrt{t/n} - \frac{\sigma^2 t}{2n}\right) \\ &\approx (H_n - T_n)\sigma\sqrt{t/n} - (H_n + T_n)\frac{\sigma^2 t}{2n} + O(n^{1/2}) \\ &\approx \frac{H_n - T_n}{\sqrt{n}}\sigma\sqrt{t} - \frac{\sigma^2 t}{2} + O(n^{1/2}) \end{aligned} \tag{16}$$

and because

$$\lim_{n\to\infty} \frac{H_n - T_n}{\sqrt{n}} \to \mathcal{N}(0,1), \tag{17}$$

when we take the $n \to \infty$ limit,

$$\begin{aligned} \ln\left(\frac{S_t}{S_0}\right) &\sim \sigma\sqrt{t}\mathcal{N}(0,1) - \frac{\sigma^2 t}{2} \\ \implies \exp\left[\lim_{n\to\infty} \ln\left(\frac{S_t}{S_0}\right)\right] &= \exp\left[-\frac{\sigma^2 t}{2} + \sigma\sqrt{t}\mathcal{N}(0,1)\right] \end{aligned} \tag{18}$$

$$\implies \boxed{S_t = S_0 e^{-\sigma^2 t/2 + \sigma\sqrt{t}\mathcal{N}(0,1)}} \tag{19}$$

which is the risk-free geometric Brownian motion stock path model.

A common modeling strategy involves splitting a time interval $[0, t]$ into n steps, so that the stock path model becomes

$$S_t = S_0 e^{-\sigma^2 t/2n + \sigma\sqrt{t/n}\mathcal{N}(0,1)}. \tag{20}$$

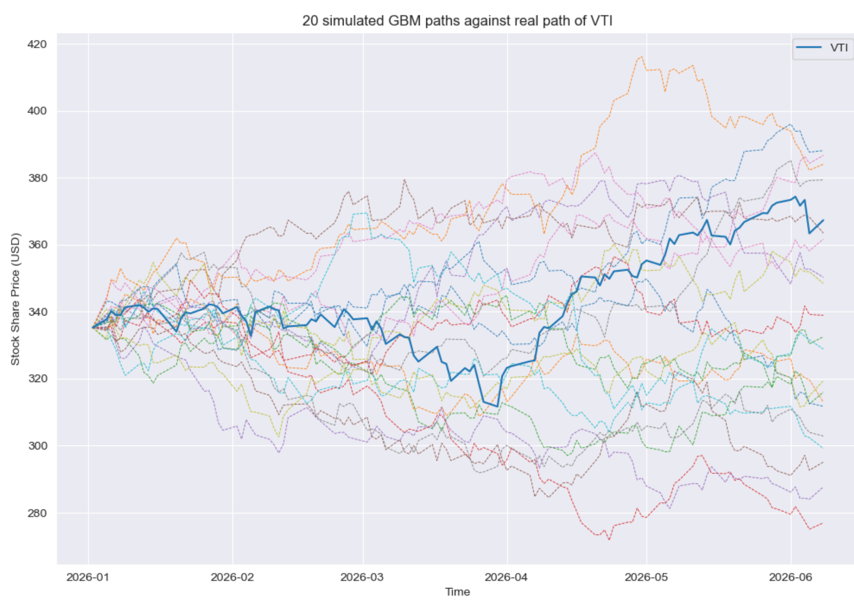

Figure 1: 20 geometric Brownian motion paths using the volatility σ and initial price S_0 for the VTI index fund against its true path over 2026.

To make this model more realistic, we add a drift term μ, which represents a stock's momentum, which could be due to macroeconomic effects such as economic growth, industry trends, and the company's performance, and an interest rate term $r \geq 0$, which accounts for the fact that just investing money into a brokerage account will accumulate interest.

The drift term's compounding effect on the stock price, either up or down depending on its sign is added to the original $n-$step geometric Brownian motion model as an additional exponential term. This changes expectation values of our model, discounted by the interest rate r, to be

$$e^{-rt}\mathbb{E}[S_t] = S_0 e^{\mu t}. \tag{21}$$

The drift term is typically not included in these models, since there are measures taken to remove these influences in when constructing trading strategies, called *delta hedging*.

The interest rate will allow us to compare stock values at later times to accurately determine by how much the *value of the stock* has increased on its own. If we start with a stock price S_0 and let the interest accumulate over $n \to \infty$ compounds,

$$S_t = \lim_{n\to\infty}\left(S_0 + \frac{S_0 rt}{n}\right) \equiv S_0 e^{rt}. \tag{22}$$

In our original $n-$step geometric Brownian motion model, $\mathbb{E}[S_t] = S_0$. When we add the interest rate term,

$$S_t = S_0 e^{(r-\sigma^2/2n)t+\sigma\sqrt{t/n}\mathcal{N}(0,1)}., \tag{23}$$

we get $\mathbb{E}[S_t] = S_0 e^{rt}$.

Note that our model is defined as being *risk-free* relative to an interest rate r since because if you average over multiple final stock prices S_t and discount the interest rate, you should get your initial investment S_0 back.

3 European Options and Black-Scholes Pricing

3.1 European Options Definitions

Definition 6. A **European call option** on a stock is a contract between the buyer and seller which gives the buyer the *right, but not the obligation*, to purchase at least one hundred shares of a single stock from the seller at a predetermined price k, called the *strike price* at a future point in time t, called the *expiration date*.

Note that the only difference between an option and a *future* is the obligation to carry out to purchase described in the contract. An option does not *require* a buyer to carry out the trade described in the contract, while a future *obligates* the buyer to carry out the trade before the contract's expiration date.

If the current stock price $S_t > k$, the buyer of an option or future will typically invoke the contract to buy the stock shares at the strike point k, immediately sell at S_t, and profit $nS_t - k$ where $n \geq 100$ is the number of shares that are included in the option or future.

Definition 7. In contrast to a call option, a **European put option** gives the buyer of the contract to *sell* a stock at a predetermined price in the future. Buying a put option is thus to short the stock.

Definition 8. An **American option** (call or put) operates in the same way as European options, but gives the buyer the right to invoke the contract at *any time before the expiration date*, whereas a buyer can only invoke the option contract *at* the expiration date for a European option.

We will start by modeling the pricing of European options and will later consider American option modeling since they add a layer of complexity.

Note that statistical methods in finance are typically reserved for option and futures trading rather than modeling individual stock prices since there is more data to take as inputs to models beyond just the price of the stock. In option and futures trading, generally called **derivatives trading** because there is no inherent value of these contracts, and is instead *derived* from the performance of the underlying asset, such as the strike price, expiration date, ask prices, and bid prices.

3.2 Black-Scholes Pricing

Definition 9. Let $t = 0$ be the current time, t be the expiration of a call option, and K be the strike price. The **Black-Scholes price of a call option** at the current time, denoted as C_0, is the expected value of the difference between a stock at expiration and the strike price, discounted by the risk-free interest rate at the expiration time:

$$C_0 = \mathbb{E}[\max(S_t - K, 0)]\, e^{-rt}, \tag{24}$$

where the $\max(\cdot)$ function between $S_t - K$ and 0 is taken since a buyer is not obligated to buy the underlying stock at the strike price K when the expiration date comes.

Replacing our known expression for S_t, our geometric Brownian motion model:

$$C_0 = \mathbb{E}\left[\max\left(S_0 e^{(r-\sigma^2/2n)t+\sigma\sqrt{t/n}\mathcal{N}(0,1)} - K, 0\right)\right] e^{-rt} \tag{25}$$

In contrast, the Black-Scholes price of a put option will be the expected value of the difference between the strike price and S_t:

$$P_0 = \mathbb{E}[\max(K - S_t, 0)]\, e^{-rt}. \tag{26}$$

We will now derive the close-form solution of C_0 by computing the expected value of the argument in Eq. (25) and then use "call-put parity" to arrive at the equivalent result for the Black-Scholes pricing of a put option.

To derive the Black-Scholes price of a call option, we fist define the cumulative distribution function:

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy \equiv \text{Prob}(\mathcal{N}(0,1) \leq x). \tag{27}$$

Assuming S_t follows a geometric Brownian motion model,

$$\begin{aligned} C_0 &\propto \mathbb{E}\left[\max\left(S_0 e^{(r-\sigma^2/2n)t+\sigma\sqrt{t/n}\mathcal{N}(0,1)} - K, 0\right)\right] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \max\left(\left(S_0 e^{(r-\sigma^2/2)t+\sigma\sqrt{t}x} - K\right) e^{-x^2/2}, 0\right) dx. \end{aligned} \tag{28}$$

We wish to find the solution where $S_t - K \geq 0$, since C_0 will otherwise vanish. Solving for the x-value that makes this true:

$$x \geq \frac{(r - \sigma^2/2)t - \ln\big(K/S_0\big)}{\sigma\sqrt{t}} := -d_2 \tag{29}$$

where we have defined d_2 so that we know the region $-d_2 \leq x < \infty$ is where the expected return stock price S_t will be greater than K (where money is made). Continuing by replacing our lower bound with $-d_2$ and removing the $\max(\cdot)$ function:

$$\begin{aligned}\mathbb{E}[\max(S_t - K, 0)] &= \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} \left(S_0 e^{(r-\sigma^2/2)t+\sigma\sqrt{t}x} - K \right) e^{-x^2/2} dx \\ &= \frac{S_0}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{(r-\sigma^2/2)t+\sigma\sqrt{t}x-x^2/2} dx - \frac{K}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-x^2/2} dx \\ &\frac{S_0 e^{rt}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-(x^2-2\sigma\sqrt{t}x+\sigma^2 t)/2} dx - K\Phi(d_2) \\ &= \frac{S_0 e^{rt}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-(x-\sigma\sqrt{t})^2/2} dx - K\Phi(d_2) \\ &= \frac{S_0 e^{rt}}{\sqrt{2\pi}} \int_{-d_2-\sigma\sqrt{t}}^{\infty} e^{-u^2/2} du - K\Phi(d_2) \\ &\equiv \frac{S_0 e^{rt}}{\sqrt{2\pi}} \int_{d_1}^{\infty} e^{-u^2/2} du - K\Phi(d_2) \\ &= \frac{S_0 e^{rt}}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-u^2/2} du - K\Phi(d_2) \\ &\equiv S_0 e^{rt}\Phi(d_1) - K\Phi(d_2),\end{aligned} \tag{30}$$

where we defined $d_1 = d_2 + \sigma\sqrt{t}$ and used the fact that a Gaussian function is even to swap the bounds on the final integral. Discounting the result of the expectation value by the interest rate gives us the Black-Scholes price of a call option:

$$\boxed{C_0 = S_0\Phi(d_1) - Ke^{-rt}\Phi(d_2)}. \tag{31}$$

We now use the so-called call-put parity to derive the Black-Scholes price of a put option, rather than carrying out a similar integration to find the expectation value of $\max(K - S_t, 0)$. First notice that if we take the difference between the value of a call and put option, the difference will always be $S_t - K$ for any S_t value:

$$\max(S_t - K, 0) - \max(K - S_t, 0) = S_t - K \;\forall\; S_t. \tag{32}$$

So if we apply the expectation value operator and discount by interest rate on both sides of this expression, we will be able to use our call option result to derive the put option price.

$$e^{-rt}\mathbb{E}[\max(S_t - K, 0)] - e^{-rt}\mathbb{E}[\max(K - S_t, 0)] = e^{-rt}\Big(\mathbb{E}[S_t] - \mathbb{E}[K]\Big)$$
$$\implies C_0 - P_0 = e^{-rt}\Big(S_0 e^{rt} - K\Big) = S_0 - Ke^{-rt}. \tag{33}$$

: since we expect to get the original investment back scaled by the interest rate for a geometric Brownian motion stock path model, $\mathbb{E}[S_t] = S_0 e^{rt}$ and the expectation value of a constant K is the constant itself.

The last line is the call-put parity:

$$\boxed{C_0 - P_0 = S_0 - Ke^{-rt}}. \tag{34}$$

Solving for P_0 and using our derived expression for C_0:

$$\begin{aligned} P_0 &= C_0 - S_0 + Ke^{-rt} \\ &= S_0\Big(1 - \Phi(d_1)\Big) + e^{rt}K\Big(1 - \Phi(d_2)\Big), \end{aligned} \tag{35}$$

and since

$$\begin{aligned} 1 - \int_{-\infty}^{d_1} xf(x)dx &= \int_{-d_1}^{\infty} xf(x)dx \\ \implies 1 - \Phi(d_1) &= \Phi(-d_1), \end{aligned} \tag{36}$$

$$\boxed{P_0 = S_0\Phi(-d_1) + e^{rt}K\Phi(-d_2)}. \tag{37}$$

Note that it makes intuitive sense that the scaling term e^{rt} should appear since you wish to compare a strike point price at the initial time to the expiration time t.

3.3 Black-Scholes Greeks

Definition 10. The **Black-Scholes Greeks** are derivatives of the Black-Scholes put and call price expressions with respect to their free variables; the initial stock price S_0, the volatility σ, expiration date t, and interest rate r.

First note that the derivative of a CDF results in the PDF by the fundamental theorem of calculus:

$$\frac{d}{dx}\Phi(x) = \frac{d}{dx}\int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}dy = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \equiv \phi(x). \tag{38}$$

Also for use in the derivation of the Greeks,

$$\frac{\phi(d_1)}{\phi(d_2)} = \frac{e^{-rt}K}{S_0}, \tag{39}$$

which follows from using the definition of the PDF $\phi(x)$ and the previously-derived relation between d_1 and d_2:

$$\boxed{d_2 = d_1 - \sigma\sqrt{t}} \tag{40}$$

$$\& \boxed{d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + (r + \sigma^2/2)t}{\sigma\sqrt{t}}}. \tag{41}$$

We can therefore write

$$S_0\phi(d_1) = e^{-rt}K\phi(d_2). \tag{42}$$

3.3.1 First-Order Greeks

We now compute the first-order Black-Scholes Greeks.

Definition 11. The **Delta** of an option contract is the rate of change of the call and put prices with respect to the initial stock price S_0:

$$\boxed{\Delta_{C_0} = \frac{\partial C_0}{\partial S_0} \quad \& \quad \Delta_{P_0} = \frac{\partial P_0}{\partial S_0}}. \tag{43}$$

Carrying these computations out:

$$\begin{aligned} \Delta_{C_0} &= \frac{\partial}{\partial S_0}\left(S_0\Phi(d_1) - e^{-rt}K\Phi(d_2)\right) \\ &= \Phi(d_1) + S_0\frac{\partial\Phi(d_1)}{\partial S_0} - e^{-rt}K\frac{\partial\Phi(d_2)}{\partial S_0} \\ &= \Phi(d_1) + S_0\phi(d_1)\frac{\partial d_1}{\partial S_0} - e^{-rt}K\phi(d_2)\frac{\partial d_2}{\partial S_0}. \end{aligned} \tag{44}$$

Since

$$\begin{aligned} d_2 &= d_1 - \sigma\sqrt{t} \\ \implies \frac{\partial d_2}{\partial S_0} &= \frac{\partial d_1}{\partial S_0} \end{aligned} \tag{45}$$

and

$$\frac{\partial d_1}{\partial S_0} = \frac{\partial}{\partial S_0}\left(\frac{\ln(S_0/K) + (r + \sigma^2/2)t}{\sigma\sqrt{t}}\right) = \frac{1}{S_0\sigma\sqrt{t}}, \tag{46}$$

we have

$$\Delta_{C_0} = \Phi(d_1) + \frac{S_0\phi(d_1)}{S_0\sigma\sqrt{t}} - e^{-rt}K\frac{\phi(d_2)}{S_0\sigma\sqrt{t}} \tag{47}$$

Using Eq. (42) for $\phi(d_2)$:

$$\phi(d_2) = \frac{S_0\phi(d_1)e^{rt}}{K} \tag{48}$$

$$\implies \Delta_{C_0} = \Phi(d_1) + \frac{S_0\phi(d_1)}{S_0\sigma\sqrt{t}} - e^{-rt}K\frac{S_0\phi(d_1)e^{rt}K}{S_0\sigma\sqrt{t}} \tag{49}$$

$$\implies \boxed{\Delta_{C_0} = \Phi(d_1)} \tag{50}$$

Using the call-put parity, written in Eq. (34), we can differentiate both sides with respect to the stock price S_0 to determine Δ_{P_0} from Δ_{C_0}:

$$\begin{aligned} \frac{\partial}{\partial S_0}\left(C_0 - P_0\right) &= \frac{\partial}{\partial S_0}\left(S_0 - Ke^{-rt}\right) \\ \Delta_{C_0} - \Delta_{P_0} = 1 &\implies \boxed{\Delta_{P_0} = \Delta_{C_0} - 1} \end{aligned} \tag{51}$$

Definition 12. The **Vega** of an option contract is the rate of chnage of the call and put prices with respect to the stock volatility σ:

$$\boxed{\nu_{C_0} = \frac{\partial C_0}{\partial \sigma} \quad \& \quad \nu_{P_0} = \frac{\partial P_0}{\partial \sigma}}. \tag{52}$$

First notice that when using call-put parity,

$$\begin{aligned} \frac{\partial}{\partial \sigma}\left(C_0 - P_0\right) &= \frac{\partial}{\partial \sigma}\left(S_0 - e^{-rt}K\right) \\ \implies \nu_{C_0} - \nu_{P_0} = 0 &\implies \nu_{C_0} = \nu_{P_0} \equiv \nu. \end{aligned} \tag{53}$$

Computing vega (for both calls and put at the same time since we have shown they are equivalent):

$$\nu = \frac{\partial C_0}{\partial \sigma} = \frac{\partial}{\partial \sigma}\Bigg(S_0\Phi(d_1) - e^{-rt}K\Phi(d_2)\Bigg) \tag{54}$$

$$= S_0\phi(d_1)\frac{\partial d_1}{\partial d\sigma} - e^{-rt}K\phi(d_2)\frac{\partial d_2}{\partial \sigma} \tag{55}$$

$$= S_0\phi(d_1)\frac{\partial d_1}{\partial d\sigma} - e^{-rt}K\phi(d_2)\Bigg(\frac{\partial d_1}{\partial \sigma} - \sqrt{t}\Bigg) \tag{56}$$

$$= S_0\phi(d_1)\frac{\partial d_1}{\partial d\sigma} - e^{-rt}K\frac{S_0\phi(d_1)e^{rt}}{K}\Bigg(\frac{\partial d_1}{\partial \sigma} - \sqrt{t}\Bigg) \tag{57}$$

$$= S_0\phi(d_1)\frac{\partial d_1}{\partial d\sigma} - S_0\phi(d_1)\Bigg(\frac{\partial d_1}{\partial \sigma} - \sqrt{t}\Bigg) \tag{58}$$

$$= \boxed{S_0\sqrt{t}\phi(d_1) = \nu} \tag{59}$$

Definition 13. The **Theta** of an option contract is the rate of change of the call and put prices with respect to the time to expiration t:

$$\boxed{\Theta_{C_0} = -\frac{\partial C_0}{\partial t} \quad \& \quad \Theta_{P_0} = -\frac{\partial P_0}{\partial t}}, \tag{60}$$

where the negative signs have been added to account for the fact that we are considering moving from a future time back to the present $t \to 0$, rather than the present to a future time.

Skipping the derivation for these since they involve the same calculus and substitution tricks as the Δ and ν derivations, the results for theta of an call and put options are:

$$\Theta_{C_0} = -\frac{S_0\sigma\Phi(d_1)}{2\sqrt{t}} - re^{-rt}K\Phi(d_2) \tag{61}$$

$$\Theta_{P_0} = -\frac{S_0\sigma\Phi(d_1)}{2\sqrt{t}} + re^{-rt}K\Phi(-d_2). \tag{62}$$

Definition 14. The **Rho** of an option contract is the rate of change of the call and put prices with respect to the interest rate r:

$$\boxed{\rho_{C_0} = \frac{\partial C_0}{\partial r} \quad \& \quad \rho_{P_0} = \frac{\partial P_0}{\partial r}}, \tag{63}$$

which won't receive much consideration since options trading typically occurs on timescales too short for an interest rate to change. These quantities would be important to consider for someone dealing with the bond market or long-term trading.

3.3.2 Second-Order Greeks

Definition 15. The **Gamma** of an option contract is the rate of change of the Δ measure with respect to the initial stock price S_0:

$$\boxed{\Gamma_{C_0} = \frac{\partial \Delta_{C_0}}{\partial S_0} \quad \& \quad \Gamma_{P_0} = \frac{\partial \Delta_{P_0}}{\partial S_0}}, \tag{64}$$

The call-put parity tells us that $\Gamma_{C_0} = \Gamma_{P_0}$ since $\partial^2_{S_0}(S_0 - e^{-rt}K) = 0$, so computing:

$$\Gamma = \frac{\partial^2 C_0}{\partial S_0^2} = \frac{\partial}{\partial S_0}\left(\Phi(d_1)\right) = \phi(d_1)\frac{\partial d_1}{\partial S_0} = \frac{\phi(d_1)}{S_0 \sigma \sqrt{t}}. \tag{65}$$

Definition 16. The **Volga** of an option contract is the rate of change of the Vega ν measure with respect to the volatility σ:

$$\boxed{\text{Volga}_{C_0} = \frac{\partial \nu_{C_0}}{\partial \sigma} \quad \& \quad \text{Volga}_{P_0} = \frac{\partial \nu_{P_0}}{\partial \sigma}}, \tag{66}$$

Skipping the derivation for these, the result is

$$\text{Volga}_{C_0} = \text{Volga}_{P_0} \equiv \text{Volga} = \frac{S_0 \sqrt{t} \phi(d_1) d_1 d_2}{\sigma} = \frac{\nu d_1 d_2}{\sigma}. \tag{67}$$

4 Delta Hedging